SYMBOLIC COMPUTATIONS OF FIRST INTEGRALS FOR POLYNOMIAL VECTOR FIELDS

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Abstract. In this article we show how to generalize to the Darbouxian, Liouvillian and Riccati case the extactic curve introduced by J. Pereira. With this approach, we get new algorithms for computing, if it exists, a rational, Darbouxian, Liouvillian or Riccati first integral with bounded degree of a polynomial planar vector field. We give probabilistic and deterministic algorithms. The arithmetic complexity of our probabilistic algorithm is in $\tilde{\mathcal{O}}(N^{\omega+1})$, where N is the bound on the degree of a representation of the first integral and $\omega \in [2;3]$ is the exponent of linear algebra. This result improves previous algorithms. Our algorithms have been implemented in Maple and are available on authors' websites. In the last section, we give some examples showing the efficiency of these algorithms. First integrals and Symbolic computations and Complexity analysis

1. Introduction

In this article, we design an algorithm that given a planar polynomial vector field with degree d

$$(S)\ :\quad \begin{cases} \dot{x} &= A(x,y),\\ \dot{y} &= B(x,y),\end{cases}\qquad A,B\in\mathbb{K}[x,y],\qquad \deg(A),\deg(B)\le d$$

and some bound $N \in \mathbb{N}$, computes first integrals of (S) of "size" (for some appropriate definition) lower than N.
The field $\mathbb{K}$ is an effective field of characteristic zero, i.e. one can perform arithmetic operations and test equality of two elements (typically, $\mathbb{K} = \mathbb{Q}$ or $\mathbb{Q}(\alpha)$, where α is an algebraic number).

First integrals are *non-constant functions* $\mathcal{F}$ that are constant along the solutions $\big(x(t), y(t)\big)$ of (S). This property can be rewritten as being solution of a partial differential equation

$$\text{(Eq)}\qquad A(x,y)\partial_x\mathcal{F}(x,y) + B(x,y)\partial_y\mathcal{F}(x,y) = 0,$$

which can be also written $D_0(\mathcal{F}) = 0$ with D_0 the derivation

$$D_0 = A(x,y)\partial_x + B(x,y)\partial_y.$$

Let us remark that multiplying A, B by some arbitrary non zero polynomial does not change the solutions of equation (Eq). Thus in the rest of the article, we will always consider $\gcd(A,B) = 1$, thus excluding the case $A = 0$ or $B = 0$ as a trivial one. Indeed, in this case $\mathcal{F}(x,y) = x$ or y is then a first integral.

We need to specify in which class of functions we are searching $\mathcal{F}$. The simplest class is the class of rational first integrals, for which we can easily define the notion of size by the degree of its numerator and denominator.

However, we can compute first integrals in larger classes of functions.
The first method for computing first integrals in a symbolic way can be credited to G. Darboux in 1878, see [15]. Darboux has introduced what we call nowadays Darboux's polynomials. These polynomials allows one to compute "Darbouxian" first integrals. Darbouxian functions are written with rational functions and logarithms and thus generalize rational functions. There exist even more general functions than Darbouxian functions, for example we can consider elementary or Liouvillian functions. There exist theoretical results about these kinds of first integrals, see [31, 36], and also algorithms for computing these first integrals, see [27, 28, 2]. Roughly speaking, these algorithms compute Darboux polynomials and then combine them in order to construct a first integral. Unfortunately, the computation of Darboux polynomials is a difficult problem. Indeed, computing a bound on the degree of the irreducible Darboux polynomials of a derivation is still an open problem. This problem is called Poincaré's problem. Thus in practice, algorithms ask the users for a bound on the "size" of the first integral they want to compute.
Moreover, the recombination step leads to an exponential complexity in terms of the degree d of the derivation.

In this article, we give an algorithm which computes a symbolic first integral: rational, Darbouxian, Liouvillian or Riccati, see Definition 1 below, with "size" bounded by N, with $\tilde{\mathcal{O}}(N^{\omega+1})$ arithmetic operations in $\mathbb{K}$.

We recall that $\omega \in [2;3]$ is the exponent of linear algebra over $\mathbb{K}$. This means that we assume that two matrices of size $N \times N$ with entries in $\mathbb{K}$ can be multiplied with $\mathcal{O}(N^\omega)$ arithmetic operations. The soft-O notation $\tilde{\mathcal{O}}()$ indicates that polylogarithmic factors are neglected. Furthermore, in the following we suppose that the bound N tends to infinity and d is fixed. The dependence in d is also polynomial. More precisely, the cost is at most $N^{\omega+1} + (d+N)^{\omega+1} + d^2N^2$ up to a constant factor and logarithmic factors in N. In particular, it shows that our algorithm is polynomial in d.
Our algorithm is thus more efficient than the existing ones.

Our strategy generalizes to the Darbouxian, Liouvillian and Riccati cases the algorithm proposed in [5] for computing rational first integrals. Our method avoids the computation of Darboux polynomials and then does not need a recombination step.

Now, we recall the definition of Darbouxian, Liouvillian first integrals and introduce a new definition: Riccati first integrals.

Definition 1.

- *A Darbouxian function $\mathcal{F}$ is an expression of the form*

$$\mathcal{F}(x,y) = \int G(x,y)dx + F(x,y)dy$$

where $F, G \in \overline{\mathbb{K}}(x,y)$ *and* $G(x,y)dx + F(x,y)dy$ *is closed, i.e.* $\partial_y G = \partial_x F$, *or equivalently*

$$\mathcal{F}(x,y) = \frac{P(x,y)}{Q(x,y)} + \sum_i \lambda_i \ln H_i(x,y)$$

where $P, Q, H_i \in \overline{\mathbb{K}}[x,y]$, $\lambda_i \in \overline{\mathbb{K}}$.

- *A Liouvillian function* $\mathcal{F}$ *is an expression of the form*

$$\mathcal{F}(x,y) = \int R(x,y)B(x,y)dx - R(x,y)A(x,y)dy$$

  where:
  - ⋆ $R(x,y) = \exp \int G(x,y)dx + F(x,y)dy$ *(called the integrating factor),*
  - ⋆ $F(x,y)$, $G(x,y)$ *belong to* $\overline{\mathbb{K}}(x,y)$,
  - ⋆ $G(x,y)dx + F(x,y)dy$ *and* $R(x,y)B(x,y)dx - R(x,y)A(x,y)dy$ *are closed.*

- *A Riccati function is an expression of the form* $\mathcal{F}_1/\mathcal{F}_2$ *where* $\mathcal{F}_1, \mathcal{F}_2$ *are two independent solutions over* $\overline{\mathbb{K}(x)}$ *of a second order differential equation*

$$(EqR) \quad \partial_y^2 \mathcal{F}_i + G(x,y)\partial_y \mathcal{F}_i + F(x,y)\mathcal{F}_i = 0$$

  with $F, G \in \overline{\mathbb{K}}(x,y)$.

- *A Darbouxian (respectively Liouvillian, or Riccati) first integral of* (S) *is a Darbouxian (respectively Liouvillian, or Riccati) function* $\mathcal{F}$ *satisfying* $D_0(\mathcal{F}) = 0$.

The classical result of the equivalence of the two representations of a Darbouxian first integral is proved in [30, 13, 17], [33, Satz 2], and in [34, Lemma 2 p. 205]. Singer [36] proves that a vector field admitting a first integral built by successive integrations, exponentiations and algebraic extensions of $\mathbb{K}(x,y)$ (so a Liouvillian function), also admits a Liouvillian first integral of the form given in Definition 1. Similarly, we will prove in Proposition 14 that a vector field admitting a first integral built by successive integrations, exponentiations, algebraic and Riccati extensions of $\mathbb{K}(x,y)$ (see Definition 13 in Section 2), also admits a Riccati first integral of the form given in Definition 1.

A vector field admitting a first integral built by successive integrations and algebraic extensions of $\mathbb{K}(x,y)$ does not always admits a Darbouxian first integral of the form given in Definition 1. This is due to the possible appearance of algebraic extensions in the 1-form $G(x,y)dx + F(x,y)dy$. However, as we will prove in Proposition 14, such vector field then admits what we call a k-Darbouxian first integral.

Definition 2. *A* k*-Darbouxian first integral of* (S) *is a first integral* $\mathcal{F}$ *of* (S) *of the form*

$$\mathcal{F}(x,y) = \int G(x,y)dx + F(x,y)dy$$

where $k \in \mathbb{N}^*$, $F^k, G^k \in \overline{\mathbb{K}}(x,y)$ *and* $G(x,y)dx + F(x,y)dy$ *is closed.*

Putting $k = 1$ recovers the classical Darbouxian first integrals.
Using that $\mathcal{F}$ is a first integral and so $D_0(\mathcal{F}) = 0$, we have moreover

$$\frac{G(x,y)}{B(x,y)} = -\frac{F(x,y)}{A(x,y)},$$

and this defines a hyperexponential function $R(x,y)$. Now writing $\mathcal{F}(x,y) = \int R(x,y)B(x,y)dx - R(x,y)A(x,y)dy$, we recognize the form of a Liouvillian first integral. So k-Darbouxian functions define an intermediary class between Darbouxian and Liouvillian functions.

We will not consider elementary first integrals. In [31], Prelle and Singer have proved that the study of elementary first integrals can be reduced to the study of Liouvillian first integrals with an algebraic integrating factor. This meets our Definition of k-Darbouxian first integral. However, elementary first integrals require the additional condition that the 1-form can be integrated in elementary terms, and such integration problems will not be considered in this article.

Rational, k-Darbouxian and Liouvillian first integrals are particular cases of a Riccati first integral, by simply taking $\mathcal{F}_1$ as the first integral and $\mathcal{F}_2 = 1$. Indeed, a rational, k-Darbouxian or Liouvillian first integral always satisfies a second order differential equation in y. In equation (EqR), it is always possible to multiply $\mathcal{F}_1, \mathcal{F}_2$ by the same hyperexponential function, leaving unchanged the quotient. This allows one to force the Wronskian to become 1, so we can put $G = 0$. This suggests that one can represent rational, Darbouxian, Liouvillian and Riccati first integrals as solutions of a differential equation in y.

- A rational first integral is a first integral solution of

(Rat) $$\mathcal{F} - F(x,y) = 0 \qquad F \in \overline{\mathbb{K}}(x,y) \setminus \overline{\mathbb{K}}.$$

- A k-Darbouxian first integral is a first integral solution of

(D) $$\partial_y \mathcal{F} - F(x,y) = 0, \qquad F^k \in \overline{\mathbb{K}}(x,y) \setminus \{0\}.$$

- A Liouvillian first integral is a first integral solution of

(L) $$\partial_y^2 \mathcal{F} - F(x,y)\partial_y \mathcal{F} = 0 \qquad F \in \overline{\mathbb{K}}(x,y).$$

- A Riccati first integral is a first integral quotient of two independent solutions over $\overline{\mathbb{K}(x)}$ of

(Ric) $$\partial_y^2 \mathcal{F} - F(x,y)\mathcal{F} = 0, \qquad F \in \overline{\mathbb{K}}(x,y).$$

These four equations will be the four canonical equations representing respectively each type of first integral. Once one of the above equation is found, it is possible to recover the first integral by single variable integration and linear differential equation solving.

Each case is included in the next one, leading to a ranking on the classes of first integrals

$$\text{Rational} < k\text{-Darbouxian} < \text{Liouvillian} < \text{Riccati}.$$

Each type of equation can be represented by a single rational function, thus also giving us a notion of "size".

Definition 3. *The degree of a rational, Darbouxian, Liouvillian, Riccati first integral is respectively the maximum of the degree of numerator and denominator of F (or F^k in the k-Darbouxian case) in the four above equations.*

The output of our algorithm will be an equation of the form (Rat), (D), (L) or (Ric). We will see that we can always suppose F with coefficients in $\mathbb{K}$. This does not change the degree of F, see Corollaries 31, 36, 41. The main theorem of the article is the following:

Theorem 4. *Let d be the maximum of $\deg(A)$ and $\deg(B)$. The problem of finding symbolic (rational, k-Darbouxian, Liouvillian, Riccati) first integrals with degree smaller than N can be solved in a probabilistic way with at most $\tilde{\mathcal{O}}(N^{\omega+1})$ arithmetic operations in $\mathbb{K}$, and the factorization of a univariate polynomial with degree at most N and coefficients in $\mathbb{K}$.*
More precisely, there exists an algorithm with inputs A, B, $k \in \mathbb{N}^*$, a bound N, and parametrized by initial conditions $z \in \mathbb{K}^2$ such that the possible outputs are:

- *a differential equation of one of the forms* (Rat), (D), (L), (Ric) *leading to a symbolic first integral,*
- *"None" meaning that there exists no symbolic first integral with degree smaller than N,*
- *"I don't know".*

Furthermore, if z avoids the roots of a non-zero polynomial with degree $\mathcal{O}(N^4)$ then the algorithm does not return "I don't know".
Moreover, if (S) admits a symbolic first integral with degree smaller than N then the output is a differential equation with minimal degree.

The parameter k is necessary for k-Darbouxian first integrals. An equation admitting a $k \geq 2$-Darbouxian first integral also admits a Liouvillian first integral. So for the (default) input $k = 1$, we detect Darbouxian first integrals, but the possible $k \geq 2$-Darbouxian first integral could stay unnoticed or seen as a Liouvillian first integral. The reduction of such Liouvillian first integral to a k-Darbouxian first integral comes down to an integration problem, i.e. testing if the integrating factor is an algebraic function, which will not be considered in this article.

As we use the dense representation of polynomials and $\deg(F) \leq N$, the size of the output of our algorithm is in $\mathcal{O}(N^2)$. Thus our algorithm has a sub-quadratic complexity if we use linear algebra algorithms with $\omega < 3$.

Furthermore, we can say that for almost all z the algorithm detects symbolic first integrals. Indeed, we are considering fields of characteristic zero and the output "I don't know" can only appear when z is a root of a non-zero polynomial.

Repeating the probabilistic algorithm in order to avoid bad values for z provides a deterministic algorithm with a polynomial complexity:

Corollary 5. *The probabilistic algorithm can be turned into a deterministic one. The deterministic algorithm uses at most $\tilde{\mathcal{O}}(N^{\omega+9})$ arithmetic operations in $\mathbb{K}$, and $\mathcal{O}(N^8)$ factorizations of univariate polynomials with coefficients in $\mathbb{K}$ and degree at most N.*

In practice, z is chosen uniformly at random. This allows to avoid bad situations. Then we do not need to repeat the algorithm. Thus the practical timings obtained in Section 8 do not reflect this complexity result.

At last, we mention that the algorithm can even sometimes returns equations of degree higher than N. This situation can appear for example when we are looking for a Darbouxian first integral with degree smaller than N and there exists a rational first integral of degree $2N$. We give such examples in Section 8.

1.1. **Strategy, description and theoretical contributions.** In this paper we generalize the approach given in [5] for computing rational first integrals. The main idea was to compute a solution $y(x_0, y_0; x)$ as a power series in x with coefficients in $\mathbb{K}(x_0, y_0)$ of

$$(E) \ : \quad \partial_x y(x_0, y_0; x) = \frac{B\big(x, y(x_0, y_0; x)\big)}{A\big(x, y(x_0, y_0; x)\big)}, \text{ and } y(x_0, y_0; x_0) = y_0,$$

and then find a rational function $F(x, y) \in \mathbb{K}(x, y)$ such that

$$F\big(x, y(x_0, y_0; x)\big) = F(x_0, y_0).$$

The key ingredient of this approach was the following:
If we know $y(x_0, y_0; x) \mod (x - x_0)^\sigma$ with σ big enough, then we can compute F from this truncated power series.

In [5], in order to avoid computations in $\mathbb{K}(x_0, y_0)$ two solutions of (E) with random initial conditions are used and give a probabilistic and then a deterministic algorithm.

In this article, the new ingredient is the following: we consider derivatives of the flow $y(x_0, y_0; x)$ relatively to y_0. We set

$$y_1(x_0, y_0; x) = \partial_{y_0} y(x_0, y_0; x), \quad y_2(x_0, y_0; x) = \partial_{y_0}^2 y(x_0, y_0; x),$$

$$y_3(x_0, y_0; x) = \partial_{y_0}^3 y(x_0, y_0; x).$$

In the following we will sometimes omit the dependence relatively to x_0, y_0 in the notations. We will write $y(x)$ instead of $y(x_0, y_0; x)$ and $y_r(x)$ instead of $y_r(x_0, y_0; x)$, for $r = 1, 2, 3$.

With a direct computation we remark that the functions $y(x), y_1(x), y_2(x), y_3(x)$ are solutions of the differential system:

$$(S_3')\begin{cases}(S_2')\begin{cases}(S_1')\begin{cases}(S_0')\quad \partial_x y = \dfrac{B}{A}\\[2ex] \partial_x y_1 = y_1\partial_y\left(\dfrac{B}{A}\right)\end{cases}\\[2ex] \partial_x y_2 = y_2\partial_y\left(\dfrac{B}{A}\right)+y_1^2\partial_y^2\left(\dfrac{B}{A}\right)\end{cases}\\[2ex] \partial_x y_3 = y_3\partial_y\left(\dfrac{B}{A}\right)+3y_2y_1\partial_y^2\left(\dfrac{B}{A}\right)+y_1^3\partial_y^3\left(\dfrac{B}{A}\right).\end{cases}$$

The system (S_r') gives a method to compute $y(x)$ and $y_r(x)$ for $r=1,2,3$ as power series in x. We only have to solve (S_r') using the Newton method for initial condition $x=x_0, y=y_0, y_1=1, y_2=0, y_3=0$.

Now, we introduce new variables y_1, y_2 and y_3 and we define a polynomial derivation D_r which is the Lie derivative along the vector field of (S_r').

Definition 6.

- *The system* (S_1') *is associated to the derivation* D_1 *in* $\mathbb{K}[x,y,y_1]$*:*
$$D_1 = A^2\partial_x + AB\partial_y + y_1A^2\partial_y\left(\frac{B}{A}\right)\partial_{y_1}.$$
- *The system* (S_2') *is associated to the derivation* D_2 *in* $\mathbb{K}[x,y,y_1,y_2]$*:*
$$D_2 = A^3\partial_x + A^2B\partial_y + y_1A^3\partial_y\left(\frac{B}{A}\right)\partial_{y_1} + A^3\Big(y_2\partial_y\left(\frac{B}{A}\right)+y_1^2\partial_y^2\left(\frac{B}{A}\right)\Big)\partial_{y_2}.$$
- *The system* (S_3') *is associated to the derivation* D_3 *in* $\mathbb{K}[x,y,y_1,y_2,y_3]$*:*
$$\begin{aligned}D_3 \quad = \quad & A^4\partial_x + A^3B\partial_y + y_1A^4\partial_y\left(\frac{B}{A}\right)\partial_{y_1} + A^4\Big(y_2\partial_y\left(\frac{B}{A}\right)+y_1^2\partial_y^2\left(\frac{B}{A}\right)\Big)\partial_{y_2}\\ & + A^4\Big(y_3\partial_y\left(\frac{B}{A}\right)+3y_2y_1\partial_y^2\left(\frac{B}{A}\right)+y_1^3\partial_y^3\left(\frac{B}{A}\right)\Big)\partial_{y_3}.\end{aligned}$$

Let us now consider $\mathcal{F}$ a Darbouxian first integral of D_0 such that $\partial_y\mathcal{F}=F$. We have
$$\mathcal{F}\big(x,y(x)\big)=\mathcal{F}(x_0,y_0)$$
and thus the derivative relatively to y_0 of this equation gives:
$$\partial_y\mathcal{F}\big(x,y(x)\big)y_1(x)=\partial_{y_0}\mathcal{F}(x_0,y_0).$$
Therefore if $\mathcal{F}$ is a Darbouxian first integral of D_0 with $\partial_y\mathcal{F}=F$ we get
$$F\big(x,y(x)\big)y_1(x)=F(x_0,y_0),$$
where $F\in\mathbb{K}(x,y)$.

Now the computation of the rational function F comes down to solving this equation knowing $y(x), y_1(x)$ as power series. This situation is similar to the one studied for rational first integrals.
Let us remark moreover that the rational function $F(x,y)y_1$ is constant on $\big(x, y(x), y_1(x)\big)$, where the initial condition is $y(x_0) = y_0$ and $y_1(x_0) = 1$. In Section 2, we prove that $F(x,y)y_1$ is a rational first integral for (S_1') and the even more general result:

Proposition 7.

- *The system* (S) *admits a rational first integral associated to* (Rat) *if and only if* $F(x,y)$ *is a rational first integral of* (S_0').
- *The system* (S) *admits a Darbouxian first integral associated to* (D) *if and only if* $y_1F(x,y)$ *is a rational first integral of* (S_1').
- *The system* (S) *admits a Liouvillian first integral associated to* (L) *if and only if* $y_1F(x,y) + y_2/y_1$ *is a rational first integral of* (S_2').
- *The system* (S) *admits a Riccati first integral associated to* (Ric) *if and only if* $4y_1^2F(x,y) - 2y_3/y_1 + 3y_2^2/y_1^2$ *is rational a first integral of* (S_3').

This proposition means that the computation of symbolic first integrals is reduced to the computation of a rational first integral with a given structure of a differential system (S_r'). The existence and the computation of these rationals first integrals can then be done thanks to generalized extactic curves. More precisely, this can be done with linear algebra only. For example, we get this kind of result, see Section 4:

Theorem 8 (Liouvillian extactic curve Theorem)**.**
We can construct from (S) *a matrix* $\tilde{\mathcal{E}}_{D_2}^N(x_0,y_0)$ *with entries in* $\mathbb{K}[x_0,y_0]$ *such that its determinant denoted by* $\tilde{E}_{D_2}^N(x_0,y_0)$*, satisfies the following properties:*

(1) *If* $\tilde{E}_{D_2}^N(x_0,y_0) = 0$ *then the derivation* D_0 *has a Liouvillian first integral with degree smaller than* N *or a Darbouxian first integral with degree smaller than* $2N+3d-1$ *or a rational first integral with degree smaller than* $4N+8d-3$.
(2) *If* D_0 *has a rational or a Darbouxian or a Liouvillian first integral with degree smaller than* N *then* $\tilde{E}_{D_2}^N(x_0,y_0) = 0$.

We will explain in Section 4 how to obtain explicitly the matrix $\tilde{\mathcal{E}}_{D_2}^N(x_0,y_0)$ from the derivation D_2.
We call this kind of theorem an "extactic curve theorem". Indeed, the matrix $\tilde{\mathcal{E}}_{D_2}^N$ corresponds to the study of a high order of contact between a solution of the differential system (S) and a Liouvillian function. This generalizes the situation introduced by Pereira in [29] for rational first integrals.

The main step in our algorithms will be the computation of a non-trivial element in the kernel of a matrix of the previous form. From such an element we will show that we can easily construct the rational function F appearing in equations (Rat), (D), (L), (Ric). Therefore, the computation of symbolic first integrals with bounded degree is reduced to a linear algebra problem.

1.2. **Related results.** The computation of symbolic first integrals can be credited to G. Darboux in 1878, see [15]. In this paper Darboux introduced what we call nowadays *Darboux polynomials*. A polynomial $f \in \mathbb{K}[x,y]$ is a Darboux polynomial for D_0 means that f divides $D_0(f)$. Thus f is an invariant algebraic curve. Darboux has shown how to find a Darbouxian first integral thanks to a recombination of Darboux polynomials.
This approach has been generalized in order to compute elementary first integrals by Prelle and Singer in [31]. This method has been implemented and studied in [27, 28].
In [36], Singer has given a theoretical characterization of Liouvillian first integrals. This characterization is the main ingredient of the algorithm proposed by Duarte et al. in [2, 16, 17].
The interested reader can also consult the following surveys [35, 22, 18, 37] for more results about Darboux polynomials and first integrals.

Roughly speaking, all the previous algorithms proceed as follows: first compute Darboux polynomials with bounded degree and second recombine them in order to find a first integral.
These two steps correspond to two practical difficulties. The computation of Darboux polynomials with bounded degree can be performed in polynomial time, see [10]. This method is based on the so-called extactic curve introduced by Pereira in [29] and uses a number of binary operations that is polynomial in the bound N, the degree d and the logarithm of the height of A and B. Unfortunately, the arithmetic complexity of this computation is in $\mathcal{O}(N^{4\omega+4})$, see [10].
The recombination part can be solved with linear algebra if we are looking for Darbouxian first integrals. However, if we are looking for a Liouvillian first integral then the recombination step used in [2, 16, 17] uses at least 2^d arithmetics operations. Indeed, this algorithm tries to solve a family of equation. Each equation of this family is constucted from a polynomial $\prod_i f_i^{e_i}$, where f_i is a Darboux polynomial and e_i is an unknown integer. A condition on the degree of the output leads to a condition on the degree of $\prod_i f_i^{e_i}$. With this approach if D_0 has k Darboux polynomials and the bound on the degree of $\prod_i f_i^{e_i}$ is bigger than k then we have to study at least 2^k situations.
In [2, 16, 17], in order to find a Liouvillian first integral, the authors compute a Darbouxian integrating factor $\mathcal{R} = e^{P/Q}\prod_i f_i^{c_i}$. With our approach the integrating factor $\mathcal{R}$ is related to the equation (L) in the following way:

$$\frac{\partial_y \mathcal{R}}{\mathcal{R}} = F - \frac{\partial_y A}{A}.$$

Thus our bound on the degree of F corresponds to a bound on the degree of the polynomials P, Q, f_i.

In [19], Ferragut and Giacomini have proposed a method to compute rational first integrals with bounded degree. This approach does not follow the previous strategy. The idea is to computed a bivariate polynomial annihilating $y(x_0, y_0; x)$ written as a power series solution of a first order differential equation. From this polynomial we can then deduce a rational first integral if it exists. Unfortunately,

the precision needed on the power series to get a correct output was not explicitly given.
In [5], the authors have improved the Ferragut-Giacomini's method. They have given an explicit bound on the precision needed on the power series to get a rational first integral when it exists. Furthermore, the main step of this algorithm is reduced to linear algebra only. The complexity of the probabilistic algorithm is then in $\tilde{\mathcal{O}}(N^{2\omega})$. However, as remarked by G. Villard this complexity can be lowered to $\tilde{\mathcal{O}}(N^{\omega+1})$ with an application of Hermite-Padé approximation. This approach was just study as an heuristic in [5].

The algorithm proposed in this article is based on a generalization of the extactic curve and follows the idea used in [19] and [5]. We give then a uniform strategy with a uniform complexity to compute rational, Darbouxian, Liouvillian, and Riccati first integrals. Furthermore, we explain in Section 2 why our approach cannot be generalized to another class of functions.

1.3. **Structure of the paper.** In the second section of this article we prove Proposition 7, i.e. we show how the computation of a symbolic first integral can be reduced to the computation of a rational first integral of a differential system (S'_r). In the third section, we define and study extactic hypersurfaces. We give a precise statement for the following idea: if an hypersurface has a sufficiently big order of contact with a generic solution of a differential system then this order of contact is infinite. This result will be useful in our algorithm in order to compute a solution with a sufficient precision in order to construct a first integral. As a byproduct we show that the computation of a rational first integral of a derivation in $\mathbb{K}[x_1, \dots, x_n]$ can be reduced to a linear algebra problem. In Section 4, we define the Darbouxian, Liouvillian and the Riccati extactic curve. We prove that these curves allow us to characterize the existence of symbolic first integrals with bounded degree. In Section 5, we study the evaluation of the extactic curves. In particular we characterize non-generic solutions. In Section 6, we give and prove the correctness of our algorithms based on the previous results. In Section 7, we study the complexity of our algorithms. At last, in Section 8 we give some examples thanks to our implementation of these algorithms. Our implementation is freely available at:
`http://combot.perso.math.cnrs.fr/software.html`,
`https://www.math.univ-toulouse.fr/∼cheze/Programme.html`.

1.4. **Notations.** In this article, we suppose that $\gcd(A, B) = 1$.
We denote by $\mathbb{K}[x, y]_{\leq N}$ the vector space of polynomials in x, y with coefficients in $\mathbb{K}$ of total degree less than N.
We set $div = \partial_x A + \partial_y B$.

In the following, $y(x_0, y_0; x)$ will be a power series solution of

$$(E) \ : \quad \partial_x y(x_0, y_0; x) = \frac{B(x, y(x_0, y_0; x))}{A(x, y(x_0, y_0; x))}, \text{ and } y(x_0, y_0; x_0) = y_0.$$

We set

$$\begin{aligned} y_1(x_0, y_0; x) &= \partial_{y_0} y(x_0, y_0; x), \\ y_2(x_0, y_0, x) &= \partial_{y_0}^2 y(x_0, y_0; x), \\ y_3(x_0, y_0; x) &= \partial_{y_0}^3 y(x_0, y_0; x). \end{aligned}$$

In the following, we will sometimes omit the dependence relatively to x_0 and y_0. We will write $y(x)$ instead of $y(x_0, y_0; x)$ and $y_r(x)$ instead of $y_r(x_0, y_0; x)$, for $r = 1, 2, 3$.

These functions $y(x), y_1(x), y_2(x), y_3(x)$ are solutions of

$$(S_3')\left\{\begin{array}{l}(S_2')\left\{\begin{array}{l}(S_1')\left\{\begin{array}{l}(S_0')\quad \partial_x y = \dfrac{B}{A}\\[2ex] \partial_x y_1 = y_1\partial_y\left(\dfrac{B}{A}\right)\end{array}\right.\\[4ex] \partial_x y_2 = y_2\partial_y\left(\dfrac{B}{A}\right) + y_1^2\partial_y^2\left(\dfrac{B}{A}\right)\end{array}\right.\\[4ex] \partial_x y_3 = y_3\partial_y\left(\dfrac{B}{A}\right) + 3y_2y_1\partial_y^2\left(\dfrac{B}{A}\right) + y_1^3\partial_y^3\left(\dfrac{B}{A}\right),\end{array}\right.$$

with initial condition $y(x_0) = y_0$, $y_1(x_0) = y_{1,0}$, $y_2(x_0) = y_{2,0}$, $y_3(x_0) = y_{3,0}$.

2. First integrals and differential invariants

2.1. **Representation of first integrals.** Let us first prove that equations (Rat), (D), (L), (Ric) used to represent the first integrals allow one to recover them. The next proposition explains why (Rat), (D), (L), (Ric) are admissible outputs when we are looking for rational or k-Darbouxian or Liouvillian or Riccati first integrals.

Proposition 9.

- *A rational first integral is uniquely defined by equation* (Rat)*.*
- *A k-Darbouxian first integral is defined up to addition of a constant by equation* (D)*.*
- *A Liouvillian first integral is defined up to affine transformation by equation* (L)*.*
- *A Riccati first integral is defined up to homography by equation* (Ric)*.*

Proof. The rational case.
According to equation (Rat), a rational first integral is simply F, and so defined uniquely by equation (Rat).

The Darbouxian case.
A k-Darbouxian first integral $\mathcal{F}$ satisfies an equation (D). Indeed, with our definition we have $\partial_y\mathcal{F} = F$. We also know that it should satisfy the equation of first integrals. This gives

$$\partial_y\mathcal{F} = F(x, y), \qquad A(x, y)\partial_x\mathcal{F}(x, y) + B(x, y)\partial_y\mathcal{F}(x, y) = 0.$$

Thus we know the derivative of $\mathcal{F}$ with respect to x and y, and so (D) defines $\mathcal{F}$ up to an addition of a constant.

The Liouvillian case.
A Liouvillian first integral $\mathcal{F}$ satisfies an equation (L). Setting

$$R(x, y) = \partial_y\mathcal{F}(x, y),$$

equation (L) becomes

$$\frac{\partial_y R}{R} = F(x,y).$$

We now use the first integral equation (Eq), dividing it by A and differentiating with respect to y, which gives

$$\partial_x \partial_y \mathcal{F} + \partial_y \left(\frac{B}{A} \partial_y \mathcal{F} \right) = 0,$$

$$\Rightarrow \partial_x R + \partial_y \left(\frac{B}{A} R \right) = 0,$$

$$\Rightarrow \partial_x R + \partial_y \left(\frac{B}{A} \right) R + \frac{B}{A} F R = 0,$$

$$\Rightarrow \frac{\partial_x R}{R} = -\partial_y \left(\frac{B}{A} \right) - \frac{B}{A} F.$$

Therefore we know the logarithmic derivatives of R with respect to x and y, and thus we obtain R up to a multiplication by a constant. Then as we have

$$\partial_y \mathcal{F}(x,y) = R(x,y), \quad \partial_x \mathcal{F}(x,y) = -\frac{B}{A}(x,y) R(x,y)$$

we obtain $\mathcal{F}$ from R up to addition of a constant. Thus equation (L) defines $\mathcal{F}$ up to an affine transformation.

The Riccati case.
A Riccati first integral is a quotient of two solutions $\mathcal{F}_1, \mathcal{F}_2$ of equation (Ric) that are independent over $\overline{\mathbb{K}(x)}$. Knowing that the quotient is a first integral, we have moreover

$$D_0 \left(\frac{\mathcal{F}_1}{\mathcal{F}_2} \right) = \frac{1}{\mathcal{F}_2^2} \left(D_0(\mathcal{F}_1)\mathcal{F}_2 - \mathcal{F}_1 D_0(\mathcal{F}_2) \right) = 0$$

and thus

$$\frac{D_0(\mathcal{F}_1)}{\mathcal{F}_1} = \frac{D_0(\mathcal{F}_2)}{\mathcal{F}_2}$$

Let us note $\Omega = D_0(\mathcal{F}_i)/\mathcal{F}_i$ (for $i = 1$ or 2 as they are equal), the functions $\mathcal{F}_1, \mathcal{F}_2$ are solutions of the PDE system

$$\partial_y^2 \mathcal{F}_i - F(x,y)\mathcal{F}_i = 0, \quad D_0(\mathcal{F}_i) - \Omega(x,y)\mathcal{F}_i = 0. \tag{2.1}$$

Let us now consider a solution of this system.
Due to the first equation, we can write it

$$C_1(x)\mathcal{F}_1(x,y) + C_2(x)\mathcal{F}_2(x,y).$$

Now substituting this in the second equation gives

$$C_1 \big(D_0(\mathcal{F}_1) - \Omega \mathcal{F}_1 \big) + C_2 \big(D_0(\mathcal{F}_2) - \Omega \mathcal{F}_2 \big) + A \partial_x C_1 \mathcal{F}_1 + A \partial_x C_2 \mathcal{F}_2 = 0.$$

Thus

$$\partial_x C_1(x) \mathcal{F}_1(x,y) + \partial_x C_2(x) \mathcal{F}_2(x,y) = 0.$$

As the functions $\mathcal{F}_1, \mathcal{F}_2$ are independent over the functions in x, the previous equality implies that we have $\partial_x C_1(x) = 0$, $\partial_x C_2(x) = 0$, and so C_1, C_2 are constants. Thus the dimension over $\mathbb{K}$ of the vector space of solutions of (2.1) is exactly 2. So the system (2.1) defines $\mathcal{F}_1, \mathcal{F}_2$ up to a change of basis. This change of basis acts on the quotient $\mathcal{F}_1/\mathcal{F}_2$ as a homography. □

The canonical equations of the output of our algorithm thus define the first integral up to a composition by a simple single variable function.

Below, we give a necessary and sufficient criterion for ensuring that an equation (Rat), (D), (L), (Ric) leads to a first integral.

Proposition 10.

- *A solution of equation* (Rat) *is a rational first integral if and only if*
$$D_0(F) = 0, \; F \in \overline{\mathbb{K}}(x,y) \setminus \overline{\mathbb{K}}.$$
- *A solution of equation* (D) *is a* k*-Darbouxian first integral if and only if*
$$D_0(F) = -AF\partial_y(B/A), \; F^k \in \overline{\mathbb{K}}(x,y) \setminus \{0\}.$$
- *A solution of equation* (L) *is a Liouvillian first integral if and only if*
$$D_0(F) = -A\partial_y(B/A)F - A\partial_y^2(B/A), \; F \in \overline{\mathbb{K}}(x,y).$$
- *A solution of equation* (Ric) *is a Riccati first integral if and only if*
$$D_0(F) = -2A\partial_y(B/A)F + \frac{1}{2}A\partial_y^3(B/A), \; F \in \overline{\mathbb{K}}(x,y).$$

Proof. The rational case:
This is the definition of rational first integrals.

The k-Darbouxian case:
In the k-Darbouxian case, we have
$$\partial_y\mathcal{F} = F(x,y), \quad D_0(\mathcal{F}) = 0.$$
So this is equivalent to
$$\partial_y\mathcal{F} = F(x,y), \quad \partial_x\mathcal{F} = -\frac{B}{A}F.$$
So a necessary and sufficient condition for a Darbouxian first integral $\mathcal{F}$ to exist is the closed form condition,
$$\begin{aligned}\partial_x F = -\partial_y\left(\frac{B}{A}F\right) \quad &\iff \quad \partial_x F = -\partial_y\left(\frac{B}{A}\right)F - \frac{B}{A}\partial_y F \\ &\iff \quad A\partial_x F + B\partial_y F = -A\partial_y\left(\frac{B}{A}\right)F \\ &\iff \quad D_0(F) = -A\partial_y\left(\frac{B}{A}\right)F\end{aligned}$$
which gives the condition of the proposition.

The Liouvillian case:
In the Liouvillian case, the first integral $\mathcal{F}$ has to solve the PDE system
$$(\star) \quad \partial_y^2\mathcal{F} - F\partial_y\mathcal{F} = 0, \qquad (\star\star) \quad D_0(\mathcal{F}) = 0.$$
The derivative relatively to x of $(\star)$ gives:
$$(\mathcal{A}) \quad \partial_x\partial_y^2\mathcal{F} - \partial_x F\partial_y\mathcal{F} - F\partial_x\partial_y\mathcal{F} = 0.$$

The derivative relatively to y of $(\star\star)$ divided by A and then simplified thanks to $(\star)$ gives:

$$\partial_y\partial_x\mathcal{F}+\partial_y\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}+\frac{B}{A}F\partial_y\mathcal{F}=0.$$

The derivative relatively to y of the previous equality gives:

$$(\mathcal{B})\quad \partial_y^2\partial_x\mathcal{F}+\partial_y^2\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}+\partial_y\Big(\frac{B}{A}\Big)\partial_y^2\mathcal{F}+\partial_y\Big(\frac{B}{A}F\Big)\partial_y\mathcal{F}+\frac{B}{A}F\partial_y^2\mathcal{F}=0.$$

The difference $(\mathcal{A})-(\mathcal{B})$ simplified thanks to $(\star)$ gives:

$$-\partial_xF\partial_y\mathcal{F}-F\partial_x\partial_y\mathcal{F}-\partial_y^2\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}-2\partial_y\Big(\frac{B}{A}\Big)F\partial_y\mathcal{F}-\frac{B}{A}\partial_yF\partial_y\mathcal{F}-\frac{B}{A}F\partial_y^2\mathcal{F}=0.$$

The equation $(\star\star)$ implies:

$$\begin{aligned} 0 &= -\partial_xF\partial_y\mathcal{F}-F\partial_y\Big(-\frac{B}{A}\partial_y\mathcal{F}\Big)-\partial_y^2\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}-2\partial_y\Big(\frac{B}{A}\Big)F\partial_y\mathcal{F}-\frac{B}{A}\partial_yF\partial_y\mathcal{F}\\ &\quad -\frac{B}{A}F\partial_y^2\mathcal{F}\\ &= -\Big(\partial_xF+\frac{B}{A}\partial_yF+\partial_y(B/A)F+\partial_y^2(B/A)\Big)\partial_y\mathcal{F}. \end{aligned}$$

If $\partial_y\mathcal{F}=0$ then $\mathcal{F}$ only depend on x. This is impossible as this would imply $A=0$ or $\mathcal{F}$ constant. So the only possibility left is

$$\partial_xF+\frac{B}{A}\partial_yF+\partial_y(B/A)F+\partial_y^2(B/A)=0.$$

This is the condition of the proposition.

Conversely, we suppose that $D_0(F)=-A\partial_y(B/A)F-A\partial_y^2(B/A)$. We are going to prove that in this situation D_0 has a Liouvillian first integral with integrating factor $\mathfrak{R}=e^{\mathfrak{F}}$, where $\mathfrak{F}$ is the integral of a rational closed 1-form. We set

$$\Omega_1=F-\frac{\partial_yA}{A},\quad \Omega_2=\frac{-div-B\Omega_1}{A}.$$

We are going to show that $\partial_x(\Omega_1)=\partial_y(\Omega_2)$. Then this implies that there exists a Darbouxian function $\mathfrak{F}$ such that $\Omega_1=\partial_y\mathfrak{F}$ and $\Omega_2=\partial_x\mathfrak{F}$. By construction we have $A\Omega_2+B\Omega_1=-div$, and so $B\mathfrak{R}dx-A\mathfrak{R}dy$ is closed, where $\mathfrak{R}=e^{\mathfrak{F}}$. Therefore $\mathfrak{R}$ is the integrating factor of a Liouvillian first integral, and we get the desired result. Thus, now we are going to prove that $\partial_x(\Omega_1)=\partial_y(\Omega_2)$. We have:

$$\begin{aligned} \partial_x(\Omega_1)-\partial_y(\Omega_2) &= \partial_xF-\partial_x\Big(\frac{\partial_yA}{A}\Big)\\ &\quad +\partial_y\Big(\frac{div}{A}\Big)+\partial_y\Big(\frac{B}{A}F\Big)-\partial_y\Big(\frac{B}{A}\frac{\partial_yA}{A}\Big). \end{aligned}$$

As

$$-\partial_x\Big(\frac{\partial_yA}{A}\Big)+\partial_y\Big(\frac{div}{A}\Big)=\frac{\partial_y^2B}{A}-\frac{\partial_yB\partial_yA}{A^2},$$

we get

$$\begin{aligned}\partial_x(\Omega_1)-\partial_y(\Omega_2) &= \partial_x F+\frac{\partial_y^2 B}{A}-\frac{\partial_y B\partial_y A}{A^2}+\partial_y\Big(\frac{B}{A}\Big)F+\frac{B}{A}\partial_y F\\ &\quad -\partial_y\Big(\frac{B}{A}\Big)\frac{\partial_y A}{A}-\frac{B}{A}\partial_y\Big(\frac{\partial_y A}{A}\Big).\end{aligned}$$

By rearranging the terms of the equation, we get

$$\begin{aligned}\partial_x(\Omega_1)-\partial_y(\Omega_2) &= \partial_x F+\Big(\frac{B}{A}\Big)\partial_y F+\partial_y\Big(\frac{B}{A}\Big)F\\ &\quad +\frac{\partial_y^2 B}{A}-\frac{\partial_y B\partial_y A}{A^2}-\partial_y\Big(\frac{B}{A}\Big)\frac{\partial_y A}{A}-\frac{B}{A}\partial_y\Big(\frac{\partial_y A}{A}\Big).\end{aligned}$$

Now, we note that

$$\partial_y^2\Big(\frac{B}{A}\Big)=\frac{\partial_y^2 B}{A}-\frac{\partial_y B\partial_y A}{A^2}-\partial_y\Big(\frac{B}{A}\Big)\frac{\partial_y A}{A}-\frac{B}{A}\partial_y\Big(\frac{\partial_y A}{A}\Big).$$

Thus

$$\partial_x(\Omega_1)-\partial_y(\Omega_2)=\partial_x F+\Big(\frac{B}{A}\Big)\partial_y F+\partial_y\Big(\frac{B}{A}\Big)F+\partial_y^2\Big(\frac{B}{A}\Big).$$

By hypothesis the right hand side of this equation is equal to zero. This gives the desired result.

<u>*The Riccati case*</u>:
In the Riccati case, a first integral is a quotient of two functions $\mathcal{F}_1,\mathcal{F}_2$, solutions of a PDE system of the form

$$(\Diamond)\quad \partial_y^2\mathcal{F}_i-F\mathcal{F}_i=0,\qquad (\Diamond\Diamond)\quad D_0(\mathcal{F}_i)-\Omega\mathcal{F}_i=0.$$

The derivative relatively to x of $(\Diamond)$ and then simplified by $(\Diamond\Diamond)$ gives:

$$(\mathcal{C})\quad \partial_x\partial_y^2\mathcal{F}_i-\partial_x F.\mathcal{F}_i-F\Big(\frac{\Omega}{A}\mathcal{F}_i-\frac{B}{A}\partial_y\mathcal{F}_i\Big)=0.$$

The derivative relatively to y of $(\Diamond\Diamond)$ divided by A and then simplified thanks to $(\Diamond)$ gives:

$$\partial_y\partial_x\mathcal{F}_i+\partial_y\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}_i+\frac{B}{A}F\mathcal{F}_i-\partial_y\Big(\frac{\Omega}{A}\Big)\mathcal{F}_i-\frac{\Omega}{A}\partial_y\mathcal{F}_i=0.$$

The derivative relatively to y of the previous equality and simplified thanks to $(\Diamond)$ gives:

$$\begin{aligned}(\mathcal{D})\quad 0 &= \partial_y^2\partial_x\mathcal{F}_i+\partial_y^2\Big(\frac{B}{A}\Big)\partial_y\mathcal{F}_i+\partial_y\Big(\frac{B}{A}\Big)F\mathcal{F}_i+\partial_y\Big(\frac{B}{A}F\Big)\mathcal{F}_i+\frac{B}{A}F\partial_y\mathcal{F}_i\\ &\quad -\partial_y^2\Big(\frac{\Omega}{A}\Big)\mathcal{F}_i-\partial_y\Big(\frac{\Omega}{A}\Big)\partial_y\mathcal{F}_i-\partial_y\Big(\frac{\Omega}{A}\Big)\partial_y\mathcal{F}_i-\frac{\Omega}{A}F\mathcal{F}_i.\end{aligned}$$

The difference $(\mathcal{C})-(\mathcal{D})$ gives:

$$\Big(-\partial_x F-\partial_y\Big(\frac{B}{A}\Big)F-\partial_y\Big(\frac{B}{A}F\Big)+\partial_y^2\Big(\frac{\Omega}{A}\Big)\Big)\mathcal{F}_i+\Big(2\partial_y\Big(\frac{\Omega}{A}\Big)-\partial_y^2\Big(\frac{B}{A}\Big)\Big)\partial_y\mathcal{F}_i=0.$$

The functions $\mathcal{F}_1, \mathcal{F}_2$ are independent. Thus the Wronskian of $\mathcal{F}_1, \mathcal{F}_2$ in y is not 0. Therefore, $(\mathcal{F}_1, \partial_y \mathcal{F}_1)$ and $(\mathcal{F}_2, \partial_y \mathcal{F}_2)$ are linearly independent. The above linear form thus vanishes on them and is identically 0, then

$$2\partial_y \left(\frac{\Omega}{A}\right) - \partial_y^2 \left(\frac{B}{A}\right) = 0,$$

$$-\partial_x F - \frac{B}{A}\partial_y F - 2\partial_y \Big(\frac{B}{A}\Big) F + \partial_y^2 \Big(\frac{\Omega}{A}\Big) = 0.$$

Therefore, by substituting the first equation into the second, we get:

$$\partial_x F + \frac{B}{A}\partial_y F = -2\partial_y \Big(\frac{B}{A}\Big) F + \frac{1}{2}\partial_y^3 \Big(\frac{B}{A}\Big).$$

This is the condition given by the proposition.

Conversely, let us prove that if the equation

$$D_0(F) = -2A\partial_y \Big(\frac{B}{A}\Big) F + \frac{1}{2} A\partial_y^3 \Big(\frac{B}{A}\Big)$$

is satisfied, then (Ric) leads to a Riccati first integral.

Let us choose

$$\Omega = \frac{1}{2} A\partial_y \left(\frac{B}{A}\right)$$

and prove that the system

$$\partial_y^2 \mathcal{F} - F\mathcal{F} = 0, \quad D_0(\mathcal{F}) - \Omega\mathcal{F} = 0 \tag{2.2}$$

has two independent solutions. The quotient of these two solutions gives then a Riccati first integral.

The strategy used to prove the existence of these two independent solutions is the following:
We consider $\mathcal{F}_1$, $\mathcal{F}_2$ a basis of solutions of $\partial_y^2 \mathcal{F} - F(x,y)\mathcal{F} = 0$, i.e. two solutions independent over *the constant field of functions in* x. Then, we are going to prove that there exists two independent couples (E_1, E_2) of functions in x such that $E_1\mathcal{F}_1 + E_2\mathcal{F}_2$ satisfies the second equation of (2.2). This gives then two independent solutions of equation (2.2).

In order to follow our strategy, we need an explicit expression of $\partial_x \mathcal{F}_1$ and $\partial_x \mathcal{F}_2$. These expressions will be useful when we will consider the equation $D_0(\mathcal{F}) - \Omega\mathcal{F} = 0$. Differentiating in x the equation $\partial_y^2 \mathcal{F} - F\mathcal{F} = 0$ gives

$$\partial_y^2 \partial_x \mathcal{F} - \big(\partial_x F\big)\mathcal{F} - F\partial_x \mathcal{F} = 0.$$

Setting $\mathcal{G} = \partial_x \mathcal{F}$, the previous equation can be rewritten

$$(\sharp) \quad \partial_y^2 \mathcal{G} - F\mathcal{G} = \big(\partial_x F\big)\mathcal{F}.$$

This is a linear differential equation with a non homogeneous term $\big(\partial_x F\big)\mathcal{F}$.
Now, we solve equation $(\sharp)$ with $\mathcal{F} = \mathcal{F}_i,\ i = 1, 2$, this gives:

$$(\sharp\sharp) \quad \partial_y^2 \mathcal{G}_i - F\mathcal{G}_i = \big(\partial_x F\big)\mathcal{F}_i(x,y).$$

We already know a basis of solutions of the homogeneous part, and we guess as particular solution

$$\mathcal{G}_i = -\frac{B}{A}\partial_y \mathcal{F}_i + \frac{1}{2}\mathcal{F}_i \partial_y \left(\frac{B}{A}\right).$$

Indeed, thanks to the relation $\partial_y^2 \mathcal{F} - F\mathcal{F} = 0$, we get:

$$\partial_y \mathcal{G}_i = -\frac{1}{2}\partial_y \Big(\frac{B}{A}\Big)\partial_y \mathcal{F}_i - \frac{BF}{A}\mathcal{F}_i + \frac{1}{2}\mathcal{F}_i \partial_y^2 \Big(\frac{B}{A}\Big).$$

The derivation relatively to y of the previous equation gives:

$$\partial_y^2 \mathcal{G}_i \quad = \quad -\frac{1}{2}\partial_y \Big(\frac{B}{A}\Big) F\mathcal{F}_i - \partial_y \Big(\frac{BF}{A}\Big)\mathcal{F}_i - \frac{BF}{A}\partial_y \mathcal{F}_i + \frac{1}{2}\mathcal{F}_i \partial_y^3 \Big(\frac{B}{A}\Big).$$

By rearranging the terms we get

$$\partial_y^2 \mathcal{G}_i \quad = \quad \frac{1}{2}\partial_y \Big(\frac{B}{A}\Big) F\mathcal{F}_i - \frac{BF}{A}\partial_y \mathcal{F}_i - \partial_y \Big(\frac{B}{A}\Big) F\mathcal{F}_i - \partial_y \Big(\frac{BF}{A}\Big)\partial_y \mathcal{F}_i + \frac{1}{2}\mathcal{F}_i \partial_y^3 \Big(\frac{B}{A}\Big).$$

Thanks to the definition of $\mathcal{G}_i$ we get:

$$\begin{aligned}
\partial_y^2 \mathcal{G}_i \quad &= \quad F\mathcal{G}_i + \left(-F\partial_y \Big(\frac{B}{A}\Big) - \partial_y \Big(\frac{BF}{A}\Big) + \frac{1}{2}\partial_y^3 \Big(\frac{B}{A}\Big)\right)\mathcal{F}_i \\
&= \quad F\mathcal{G}_i + \left(-2\partial_y \Big(\frac{B}{A}\Big) F - \frac{B}{A}\partial_y F + \frac{1}{2}\partial_y^3 \Big(\frac{B}{A}\Big)\right)\mathcal{F}_i.
\end{aligned}$$

As by hypothesis, we have $D_0(F) = -2A\partial_y \Big(\frac{B}{A}\Big) F + \frac{1}{2}A\partial_y^3 \Big(\frac{B}{A}\Big)$, this implies

$$-2\partial_y \Big(\frac{B}{A}\Big) F - \frac{B}{A}\partial_y F + \frac{1}{2}\partial_y^3 \Big(\frac{B}{A}\Big) = \partial_x F.$$

This proves that $\mathcal{G}_i$ is a particular solution of ($\sharp\sharp$).

So the solutions of ($\sharp\sharp$) are respectively for $i = 1, 2$ of the form

$$\mathcal{G}_1 = C_1\mathcal{F}_1 + C_2\mathcal{F}_2 - \frac{B}{A}\partial_y \mathcal{F}_1 + \frac{1}{2}\mathcal{F}_1 \partial_y \left(\frac{B}{A}\right)$$

$$\mathcal{G}_2 = C_3\mathcal{F}_1 + C_4\mathcal{F}_2 - \frac{B}{A}\partial_y \mathcal{F}_2 + \frac{1}{2}\mathcal{F}_2 \partial_y \left(\frac{B}{A}\right)$$

where the C_i *depend on x only*. So we deduce that there exists C_1, C_2, C_3, C_4 functions of x only such that

$$\partial_x \mathcal{F}_1 = C_1\mathcal{F}_1 + C_2\mathcal{F}_2 - \frac{B}{A}\partial_y \mathcal{F}_1 + \frac{1}{2}\mathcal{F}_1 \partial_y \left(\frac{B}{A}\right),$$

$$\partial_x \mathcal{F}_2 = C_3\mathcal{F}_1 + C_4\mathcal{F}_2 - \frac{B}{A}\partial_y \mathcal{F}_2 + \frac{1}{2}\mathcal{F}_2 \partial_y \left(\frac{B}{A}\right).$$

We now search solutions of equations (2.2) of the form

$$E_1\mathcal{F}_1 + E_2\mathcal{F}_2$$

with E_1, E_2 functions of x only. Substituting it in equations (2.2), we obtain 0 for the first, and for the second

$$(E_1C_1 + E_2C_3 + \partial_x E_1)\mathcal{F}_1 + (E_1C_2 + E_2C_4 + \partial_x E_2)\mathcal{F}_2 = 0.$$

As $\mathcal{F}_1, \mathcal{F}_2$ are independent over functions in x, this is equivalent to the system

$$\partial_x E_1 = -E_1 C_1 - E_2 C_3, \quad \partial_x E_2 = -E_1 C_2 - E_2 C_4.$$

This is a 2×2 linear differential system, and so admits two independent solutions. Then these two solutions E_1, E_2 give two independent solutions of equations (2.2) of the form $E_1\mathcal{F}_1 + E_2\mathcal{F}_2$. Their quotient is then a first integral. $\square$

During the previous proof we have shown the following:

Corollary 11. *If D_0 has a Riccati first integral $\mathcal{F}_1/\mathcal{F}_2$ then we can suppose that*

$$D_0(\mathcal{F}_i) = \frac{1}{2} A \partial_y \Big(\frac{B}{A}\Big) \mathcal{F}_i.$$

2.2. **Casale's Theorem.** We have defined 4 types of first integrals. We are going to prove that there are no other types of first integrals with algebraic-differential properties. Recall that the flow is defined by

$$\partial_x y(x_0, y_0; x) = \frac{B\big(x, y(x_0, y_0; x)\big)}{A\big(x, y(x_0, y_0; x)\big)}, \text{ and } y(x_0, y_0; x_0) = y_0.$$

We are also interested in $y_r(x) = \partial_{y_0}^r y(x_0, y_0; x)$, $r = 1, 2, 3$. These functions belong to $\mathbb{K}(x_0, y_0)[[x - x_0]]$. As explained in the introduction, y, y_1, y_2, y_3 are seen as functions in x, solutions of some differential systems (S'_r), and their initial conditions are

$$y(x_0) = y_0, \; y_1(x_0) = 1, \; y_2(x_0) = 0, \; y_3(x_0) = 0.$$

If $\mathcal{F}$ is a first integral of D_0 then we have $\mathcal{F}\big(x, y(x_0, y_0; x)\big) = \mathcal{F}(x_0, y_0)$. As mentioned in the introduction, the derivation relatively to y_0 of this relation gives with our notations:

$$(\star) \quad \partial_y \mathcal{F}\big(x, y(x)\big) y_1(x) = \partial_{y_0} \mathcal{F}(x_0, y_0).$$

Therefore if $\mathcal{F}$ is a Darbouxian first integral, we have $\partial_y \mathcal{F} = F \in \overline{\mathbb{K}}(x, y)$ and then

$$F\big(x, y(x)\big) y_1(x) = F(x_0, y_0).$$

Thus the rational function $F(x, y) y_1 \in \mathbb{K}(x, y, y_1)$ is constant on $\big(x, y(x), y_1(x)\big)$, where the initial condition is $y(x_0) = y_0$ and $y_1(x_0) = 1$. In Proposition 7, we prove that $F(x, y) y_1$ is also a rational first integral for (S'_1).

The derivation relatively to y_0 of equation $(\star)$ gives

$$\partial_y^2 \mathcal{F}\big(x, y(x)\big) \Big(y_1(x)\Big)^2 + \partial_y \mathcal{F}\big(x, y(x)\big) y_2(x) = \partial_{y_0}^2 \mathcal{F}(x_0, y_0).$$

If $\mathcal{F}$ is Liouvillian then $\partial_y^2 \mathcal{F} / \partial_y \mathcal{F} = F$, and we get

$$F\big(x, y(x)\big) y_1(x) + \frac{y_2(x)}{y_1(x)} = F(x_0, y_0).$$

We are also going to prove in Proposition 7 that $F(x, y) y_1 + y_2/y_1$ is a rational first integral of (S'_2).

At last, for a Riccati first integral similar computations give a rational expression in $x, y, y_1, \ldots, y_3$ which happens to be a rational first integral for (S'_3).

The reason why we stop at $r = 3$ is the following.

Theorem 12 (Casale)**.** *If there exists $J(x,y,y_1,\dots,y_n) \in \mathbb{K}(x,y,y_1,\dots,y_n)$ such that*

$$J\big(x_0,y(x_0),y_1(x_0),\dots,y_n(x_0)\big) = J\big(x,y(x),y_1(x),\dots,y_n(x)\big),$$

where $y_i(x) = \partial^i_{y_0} y(x_0,y_0;x)$, then there exists a rational function $h(x,y) \in \mathbb{K}(x,y)$ satisfying one of the following equalities:

- $h\big(x_0,y(x_0)\big) = h\big(x,y(x)\big)$,
- $h\big(x_0,y(x_0)\big) = h\big(x,y(x)\big)y_1(x)^k$, *with* $k \in \mathbb{N}^*$,
- $h\big(x_0,y(x_0)\big) = h\big(x,y(x)\big)y_1(x) + y_2(x)/y_1(x)$,
- $h\big(x_0,y(x_0)\big) = h\big(x,y(x)\big)y_1^2(x) + 2y_3(x)/y_1(x) - 3y_2^2(x)/y_1^2(x)$.

This result is Proposition 1.18 and Theorem 1.19 of Casale in [9] applied to the map $y_0 \mapsto \varphi(x_0,y_0;x)$, and restricted to the case of rational instead of meromorphic invariants. Now, Casale's invariants can be seen as first integrals of the systems (S'_r) and satisfy the equations $D_r(J)=0$.
We will associate to each class of first integral a Casale invariant for the flow. This gives the following proposition stated in the introduction:

Proposition 7.

- *The system (S) admits a rational first integral associated to* (Rat) *if and only if $F(x,y)$ is a first integral of (S'_0), where $F \in \overline{\mathbb{K}}(x,y) \setminus \mathbb{K}$.*
- *The system (S) admits a k-Darbouxian first integral associated to* (D) *if and only if $y_1F(x,y)$ is a first integral of (S'_1), where $F^k \in \overline{\mathbb{K}}(x,y) \setminus \{0\}$.*
- *The system (S) admits a Liouvillian first integral associated to* (L) *if and only if $y_1F(x,y) + y_2/y_1$ is a first integral of (S'_2), where $F \in \overline{\mathbb{K}}(x,y)$.*
- *The system (S) admits a Riccati first integral associated to* (Ric) *if and only if $4y_1^2F(x,y) - 2y_3/y_1 + 3y_2^2/y_1^2$ is a first integral of (S'_3), where $F \in \overline{\mathbb{K}}(x,y)$.*

In the Riccati case, we write our invariant in the form

$$4y_1^2F(x,y) - 2y_3/y_1 + 3y_2^2/y_1^2$$

and not in the form

$$y_1^2F(x,y) - 2y_3/y_1 + 3y_2^2/y_1^2$$

as in Casale's theorem. We have chosen this expression because it leads to the Riccati equation (Ric): $\partial_y^2\mathcal{F} - F\mathcal{F} = 0$, whereas the expression used in Casale's theorem leads to the equation $\partial_y^2\mathcal{F} - \frac{F}{4}\mathcal{F} = 0$.

Proof. <u>The rational case.</u>
By definition, a rational first integral of (S'_0) is a rational function $F(x,y)$ such that

$F(x, y(x_0, y_0; x)) = F(x_0, y_0)$. We deduce then

$$\begin{aligned} & \partial_x F\big(x, y(x_0, y_0; x)\big) + \partial_y F\big(x, y(x_0, y_0; x)\big)\frac{B}{A}\big(x, y(x_0, y_0; x)\big) = 0 \\ \iff \quad & \partial_x F(x_0, y_0) + \partial_y F\big(x_0, y_0\big)\frac{B}{A}(x_0, y_0) = 0 \\ \iff \quad & D_0(F) = 0. \end{aligned}$$

This gives the desired conclusion in the rational case.

The k-Darbouxian case.
We have: $y_1 F(x, y)$ is a first integral of (S_1') if and only if $D_1\big(y_1 F\big) = 0$, where $D_1 = A^2\partial_x + AB\partial_y + y_1 A^2 \partial_y\Big(\dfrac{B}{A}\Big)\partial_{y_1}$.
This gives:

$$\begin{aligned} D_1\big(y_1 F\big) = 0 \quad & \iff \quad y_1\left(\partial_x F + \frac{B}{A}\partial_y F + \partial_y\Big(\frac{B}{A}\Big)F\right) = 0 \\ & \iff \quad D_0(F) = -A\partial_y\Big(\frac{B}{A}\Big)F. \end{aligned}$$

Then, Proposition 10 gives the desired conclusion.

The Liouvillian case.
We have:
$y_1 F(x, y) + y_2/y_1$ is a first integral of (S_2') if and only if $D_2(y_1 F + y_2/y_1) = 0$, and we recall that
$D_2 = A^3\partial_x + A^2 B\partial_y + y_1 A^3 \partial_y\Big(\dfrac{B}{A}\Big)\partial_{y_1} + A^3\Big(y_2\partial_y\Big(\dfrac{B}{A}\Big) + y_1^2\partial_y^2\Big(\dfrac{B}{A}\Big)\Big)\partial_{y_2}$.
This gives:

$$\begin{aligned} D_2(y_1 F + y_2/y_1) = 0 \quad & \iff \quad y_1\left(\partial_x F + \frac{B}{A}\partial_y F + \partial_y\Big(\frac{B}{A}\Big)F + \partial_y^2\Big(\frac{B}{A}\Big)\right) = 0 \\ & \iff \quad D_0(F) = -A\partial_y\Big(\frac{B}{A}\Big)F - A\partial_y^2\Big(\frac{B}{A}\Big). \end{aligned}$$

As before we get the desired conclusion thanks to Proposition 10.

The Riccati case.
We consider the derivation

$$\begin{aligned} D_3 \quad = \quad & A^4\partial_x + A^3 B\partial_y + y_1 A^4\partial_y\left(\frac{B}{A}\right)\partial_{y_1} + A^4\Big(y_2\partial_y\left(\frac{B}{A}\right) + y_1^2\partial_y^2\left(\frac{B}{A}\right)\Big)\partial_{y_2} \\ & + A^4\Big(y_3\partial_y\left(\frac{B}{A}\right) + 3y_2 y_1\partial_y^2\left(\frac{B}{A}\right) + y_1^3\partial_y^3\left(\frac{B}{A}\right)\Big)\partial_{y_3}, \end{aligned}$$

and we have: $4y_1^2 F(x, y) - 2y_3/y_1 + 3y_2^2/y_1^2$ is a first integral of (S_3') if and only if $D_3(4y_1^2 F(x, y) - 2y_3/y_1 + 3y_2^2/y_1^2) = 0$.

As $D_3(-2y_3/y_1+3y_2^2/y_1^2) = -2y_1^2\partial_y^3(B/A)$, we get:

$$\begin{aligned} & D_3(4y_1^2F(x,y)-2y_3/y_1+3y_2^2/y_1^2)=0\\ \iff\quad & y_1^2\left(\partial_xF+\frac{B}{A}\partial_yF+2A\partial_y\left(\frac{B}{A}\right)F-\frac{1}{2}A\partial_y^3\left(\frac{B}{A}\right)\right)=0\\ \iff\quad & D_0(F)=-2A\partial_y\left(\frac{B}{A}\right)F+\frac{1}{2}A\partial_y^3\left(\frac{B}{A}\right). \end{aligned}$$

We conclude thanks to Proposition 10. □

Now, we recall some definitions, see [9, Definition 4.1].

Definition 13. *Let $(K;\partial_x,\partial_y)$ be a differential field. An algebraic extension $L\supset K$ is a differential field such that $L=K(f)$ with f algebraic over K. An exponential extension $L\supset K$ is a differential field such that $L=K(\exp f)$ with $f\in K$.*
A primitive extension $L\supset K$ is a differential field such that $L=K(f)$ with

$$df=\partial_xf dx+\partial_yf dy$$

a 1-form with coefficients in K, i.e. ∂_xf and ∂_yf belong to K.
A Riccati extension $L\supset K$ is a differential field such that $L=K(f)$ with df a 1-form with coefficients in $K[f]_{\leq 2}$.

Proposition 14.

- *The system (S) admits a first integral in a field built by successive algebraic and primitive extensions over $\mathbb{K}(x,y)$ if and only if it admits a k-Darbouxian first integral.*
- *The system (S) admits a first integral in a field built by successive algebraic, exponential, primitive extensions, over $\mathbb{K}(x,y)$ if and only if it admits a Liouvillian first integral.*
- *The system (S) admits a first integral in a field built by successive algebraic extensions, exponential, primitive and Riccati extensions over $\mathbb{K}(x,y)$ if and only if it admits a Riccati first integral.*

Proof. *The Darbouxian case.*
If (S) admits a first integral built by successive algebraic and primitive extensions over $\mathbb{K}(x,y)$, then by Theorem 4.2, Theorem 1.19 and Proposition 1.18 of Casale [9] there exists $k\in\mathbb{N}^*, F\in\overline{\mathbb{K}}(x,y)$ such that $y_1^kF(x,y)$ is a first integral of (S_1'). By Proposition 7, then (S) admits a k-Darbouxian first integral. The converse is immediate as a k-Darbouxian first integral is the integral of an algebraic 1-form.

The Liouvillian case.
This case corresponds to Singer's result, see [36].

The Riccati case.
Thanks to Theorem 4.2, Theorem 1.19 and Proposition 1.18 of Casale [9], if (S) admits a first integral in a field built by successive algebraic exponential, primitive and Riccati extensions over $\mathbb{K}(x,y)$, then

$$4y_1^2F(x,y)-2y_3/y_1+3y_2^2/y_1^2$$

is a first integral of (S_3'). Then, Proposition 7 implies that (S) admits a Riccati first integral.
For the converse, we have a first integral which is the quotient of two solutions

$\mathcal{F}_1, \mathcal{F}_2$ of a linear second order differential equation in y. We also know, thanks to Corollary 11, that $\mathcal{F}_1, \mathcal{F}_2$ satisfy the equation

$$D_0(\mathcal{F}) = \frac{1}{2}A\partial_y\left(\frac{B}{A}\right)\mathcal{F},$$

and thus writing $\partial_y\mathcal{F}$ as a function of $\partial_x\mathcal{F}$, we obtain another linear second order differential equation in x of the following kind: $\partial_x^2\mathcal{F} = R\partial_x\mathcal{F} + S\mathcal{F}$, where R, S belong to $\mathbb{K}(x, y)$.
Therefore, $f_{1,i} = \partial_x\mathcal{F}_i/\mathcal{F}_i$ is a solution of the following Riccati associated equation:

$$\partial_x f_{1,i} = Rf_{1,i} + S - f_{1,i}^2.$$

Thus $\partial_x f_{1,i} \in \mathbb{K}(x, y)[f_{1,i}]_{\leq 2}$.
Furthermore, $f_{2,i} = \partial_y\mathcal{F}_i/\mathcal{F}_i$ is a solution of the Riccati equation:

$$(\star) \quad \partial_y f_{2,i} = F - f_{2,i}^2.$$

Thus $\partial_y f_{2,i} \in \mathbb{K}(x, y)[f_{2,i}]_{\leq 2}$.
Now we are going to prove that $\partial_y f_{i,i} \in \mathbb{K}(x, y)[f_{1,i}]_{\leq 2}$. We have

$$(\star\star) \quad Af_{1,i} + Bf_{2,i} = \frac{1}{2}A\partial_y\left(\frac{B}{A}\right)$$

and thus, by dividing by A and derivating relatively to y the equation $(\star\star)$, we get:

$$\partial_y f_{1,i} \quad = \quad -\partial_y\left(\frac{B}{A}\right)f_{2,i} - \frac{B}{A}\partial_y f_{2,i} + \partial_y\left(\frac{1}{2}\partial_y\left(\frac{B}{A}\right)\right).$$

The equation $(\star)$ gives

$$\partial_y f_{1,i} \quad = \quad -\partial_y\left(\frac{B}{A}\right)f_{2,i} - \frac{B}{A}(F - f_{2,i}^2) + \partial_y\left(\frac{1}{2}\partial_y\left(\frac{B}{A}\right)\right).$$

Now, thanks to $(\star\star)$ we can write $f_{2,i}$ in terms of $f_{1,i}$ and this gives

$$\begin{aligned}\partial_y f_{1,i} \quad = \quad & -\partial_y\left(\frac{B}{A}\right)\left(\frac{1}{2}\frac{A}{B}\partial_y\left(\frac{B}{A}\right) - \frac{A}{B}f_{1,i}\right) - \frac{B}{A}\Big[F - \Big(\frac{1}{2}\frac{A}{B}\partial_y\left(\frac{B}{A}\right) - \frac{A}{B}f_{1,i}\Big)^2\Big] \\ & + \partial_y\left[\frac{1}{2}\partial_y\left(\frac{B}{A}\right)\right] \in \mathbb{K}(x, y)[f_{1,i}]_{\leq 2}.\end{aligned}$$

Symmetrically, we also obtain that $\partial_x f_{2,i} \in \mathbb{K}(x, y)[f_{2,i}]_{\leq 2}$. Then, we can construct a (four successive) Riccati extension L_1 of $\mathbb{K}(x, y)$ containing

$$\partial_y\mathcal{F}_1/\mathcal{F}_1, \partial_y\mathcal{F}_2/\mathcal{F}_2, \partial_x\mathcal{F}_1/\mathcal{F}_1, \partial_x\mathcal{F}_2/\mathcal{F}_2.$$

Now taking a primitive extension and then an exponential extension for each $\mathcal{F}_i$, we obtain a field L_2 containing $\mathcal{F}_1, \mathcal{F}_2$, and thus the first integral $\mathcal{F}_1/\mathcal{F}_2$. □

3. Extactic hypersurfaces

As already remarked in Proposition 7, the existence of a Darbouxian, or Liouvillian or Riccati first integral is equivalent to the existence of a rational first integral with a special structure for a derivation associated to the problem. Furthermore, in this situation we have new variables y_1, y_2, y_3. In the following we will need to study rational first integral for derivations with several variables x, y, y_1, y_2, y_3. Thus, in this section we study the characterization of rational first integrals for a derivation with variables $x, y, y_1, \ldots, y_n$. Our main tool will be the extactic curve. This curve has been discovered independently by Lagutinski and Pereira, see [29]. It allows one to characterize the situation where a derivation has a rational first integral with bounded degree. Here, we define and prove the main property of this object for a derivation in $\mathbb{K}(x, y, y_1, \ldots, y_n)$ and we will get extactic hypersurfaces. A similar study in the bivariate case has already be done in [11].

We consider a derivation

$$D = f\partial_x + f_0\partial_y + \sum_{j=1}^{n} f_j \partial_{y_j}, \text{ where } f_j \in \mathbb{K}[x, y, y_1, \ldots, y_n]$$

with $f \neq 0$ and we consider the associated differential system:

$$(S_n) \quad \begin{cases} \partial_x y(x) = \dfrac{f_0\big(x, y_1(x), \ldots, y_n(x)\big)}{f\big(x, y_1(x), \ldots, y_n(x)\big)}, \\ \partial_x y_j(x) = \dfrac{f_j\big(x, y_1(x), \ldots, y_n(x)\big)}{f\big(x, y_1(x), \ldots, y_n(x)\big)}, \text{ for } j = 1, \ldots, n. \end{cases}$$

We want to characterize the existence of a rational first integral with degree smaller than N for this kind of differential system. The idea is to study the order of contact between a solution of (S_n) and a polynomial.

Definition 15. *We set* $\underline{y}(x) = \big(y(x), y_1(x) \ldots, y_n(x)\big)$.
A parametrized curve $\big(x, \underline{y}(x)\big)$ *and an hypersurface defined by the zero locus of* $g(x, y, y_1, \ldots, y_n) \in \mathbb{K}[x, y, y_1, \ldots, y_n]$ *have a contact of order* ν *at* $(x_0, y_0, y_{1,0}, \ldots, y_{n,0})$ $= \big(x_0, \underline{y}(x_0)\big)$, *when* ν *is the biggest integer such that:*

$$g\big(x, \underline{y}(x)\big) = 0 \mod (x - x_0)^{\nu}.$$

In order to find a rational first integral for a plane vector field, it is sufficient to compute an algebraic curve with a "high enough" order of contact with a *generic solution of* (S_0'), see [5]. Actually, here "high enough" means infinite. Indeed, we will see that if the order of contact is large enough (a bound will be given later) then the order of contact will be infinite. We are going to use the same approach in the multivariate case.

In control theory this kind of problem is classical. Risler, in [32], and Gabrielov [20] have shown that if the order of contact is big enough then it is infinite. More precisely, in [20] the author shows that if $y(x)$ is a solution of a differential system with degree d and g is a polynomial with degree k such that $g(x, \underline{y}(x)) \neq 0$ then the order of contact between $g = 0$ and $(x, \underline{y}(x))$ is smaller than $2^{2n+3} \sum_{j=1}^{n+2} [k + (j-1)(d-1)]^{2n+4}$.
Here, we will get a better bound because we are going to consider a solution $y(x)$ with a *generic initial condition.*
In the following $x_0, y_0, y_{1,0}, \ldots, y_{n,0}$ *are new variables.* They correspond to a generic initial condition.

In order to compute the order of contact between g and a solution

$$\big(x, \underline{y}(x)\big) = \big(x, y(x), y_1(x), \ldots, y_n(x)\big)$$

we have to compute the Taylor expansion of $g(x,\underline{y}(x))$ at x_0. As

$$f\partial_x\Big(g\big(x,\underline{y}(x)\big)\Big) = D(g)\big(x,\underline{y}(x)\big),$$

and $f\neq 0$ we deduce easily that for all $l\in\mathbb{N}$

$$\partial_x^i g\big(x,\underline{y}(x)\big)=0,\ \text{for } i=1,\dots,l \iff D^i(g)(x_0,y_0,y_{1,0},\dots,y_{n,0})=0,\ \text{for } i=1,\dots,l.$$

where $D^0(g)=g$ and $D^i(g)=D\big(D^{i-1}(g)\big)$.

The study of the order of contact at a generic point $(x_0,y_0,y_{1,0},\dots,y_{n,0})$ leads us to consider the following map:

Definition 16. *Let D be a derivation and let V be a linear subspace of dimension l of $\mathbb{K}[x,\underline{y}]$, where $\underline{y}=(y,y_1,\dots,y_n)$.*
Let x_0 and $\underline{y}_0=(y_0,y_{1,0},\dots,y_{n,0})$ be new variables and set $\mathbb{L}=\mathbb{K}(x_0,\underline{y}_0)$. We consider the linear $\mathbb{L}$-morphism:

$$\begin{array}{rcl}\mathcal{E}_D^V:\mathbb{L}\otimes_{\mathbb{K}} V & \longrightarrow & \mathbb{L}^l\\ g(x,\underline{y}) & \longmapsto & \big(g(x_0,\underline{y}_0),D(g)(x_0,\underline{y}_0),D^2(g)(x_0,\underline{y}_0),\dots,D^{l-1}(g)(x_0,\underline{y}_0)\big)\end{array}$$

where $D^k(g)=D\big(D^{k-1}(g)\big)$ and D is, by abuse of notation, the extension of the derivation D to $\mathbb{L}[x,\underline{y}]$, i.e.

$$D\Big(\sum_\alpha c_\alpha(x_0,\underline{y}_0)x^{\alpha_1}\underline{y}^{\alpha_2}\Big)=\sum_\alpha c_\alpha(x_0,\underline{y}_0)D(x^{\alpha_1}\underline{y}^{\alpha_2}).$$

The determinant of this linear map is denoted by $E_D^V(x_0,\underline{y}_0)$.
Moreover, we call the hypersurface $E_D^{\mathbb{K}[x,\underline{y}]_{\leq N}}(x_0,\underline{y}_0)$ the N-th extactic hypersurface.

Remark 17. *With this definition we have:*

$$g(x,\underline{y})\in\ker\mathcal{E}_D^V \iff g\big(x,\underline{y}(x)\big)=0 \mod (x-x_0)^l.$$

If $\{g_1,\dots,g_l\}$ is a basis of V then the associated extactic hypersurface is given by

$$E_D^V(x_0,\underline{y}_0)=\begin{vmatrix} g_1(x_0,\underline{y}_0) & g_2(x_0,\underline{y}_0) & \dots & g_l(x_0,\underline{y}_0)\\ D(g_1)(x_0,\underline{y}_0) & D(g_2)(x_0,\underline{y}_0) & \dots & D(g_l)(x_0,\underline{y}_0)\\ \vdots & \vdots & \vdots & \vdots\\ D^{l-1}(g_1)(x_0,\underline{y}_0) & D^{l-1}(g_2)(x_0,\underline{y}_0) & \dots & D^{l-1}(g_l)(x_0,\underline{y}_0)\end{vmatrix}.$$

The extactic hypersurface is related to invariant algebraic hypersurfaces. We call these kinds of hypersurfaces Darboux polynomials.

Definition 18. *A non constant polynomial $M\in\overline{\mathbb{K}}[x,\underline{y}]$ is a* Darboux polynomial for *D if M divides $D(M)$ in $\overline{\mathbb{K}}[x,\underline{y}]$. We call the polynomial $\Lambda=D(M)/M$ the* cofactor associated with the Darboux polynomial *M.*

Proposition 19. *If $g\in V$ is a Darboux polynomial of a derivation D then $g(x_0,\underline{y}_0)$ is a factor of $E_D^V(x_0,\underline{y}_0)$.*

Proof. Let $\mathcal{B} = \{g, g_2, \dots, g_l\}$ be a basis of V.
As g is a Darboux polynomial we have $D(g) = \Lambda.g$ where $\Lambda \in \mathbb{K}[x, \underline{y}]$. Thus there exist polynomials Λ_j such that $D^j(g) = \Lambda_j.g$. Thus, we have

$$\mathcal{E}_D^V(g) = g(x_0, \underline{y}_0).\big(1, \Lambda(x_0, \underline{y}_0), \dots, \Lambda_l(x_0, \underline{y}_0)\big)$$

and $g(x_0, \underline{y}_0)$ is a factor of the first column of a matrix representation of $\mathcal{E}_D^V$ in the basis $\mathcal{B}$. It follows that g is factor of $E_D^V(x_0, \underline{y}_0)$. □

We remark that by definition the factors of $E_D^V(x_0, \underline{y}_0)$ which are not Darboux polynomials correspond to algebraic hypersurfaces with order of contact with a solution $(x, y(x), y_1(x), \dots, y_n(x))$ bigger than $l = \dim_{\mathbb{K}} V$.

We also remark that the determinant $E_D^V(x_0, \underline{y}_0)$ corresponds to a Wronskian and we recall the following classical lemma, see [8, Lemma 3.3.5]:

Lemma 20. *Let (F, D) be a differential field. Then $g_1, \dots, g_l \in F$ are linearly dependent over* $\ker D$ *if and only if $W_D(g_1, \dots, g_l) = 0$, where*

$$W_D(g_1, \dots, g_l) = \begin{vmatrix} g_1 & g_2 & \dots & g_l \\ D(g_1) & D(g_2) & \dots & D(g_l) \\ \vdots & \vdots & \vdots & \vdots \\ D^{l-1}(g_1) & D^{l-1}(g_2) & \dots & D^{l-1}(g_l) \end{vmatrix}$$

is the Wronskian of $g_1, \dots, g_l$ relatively to D.

This leads to the following proposition:

Proposition 21. *There exists a $\mathbb{K}$ vector space V such that $E_D^V(x_0, \underline{y}_0) = 0$ if and only if D has a rational first integral.*

Proof. We denote by $\{g_1, \dots, g_l\}$ a basis of V. This basis gives a basis of the $\mathbb{L}$ vector space $\mathbb{L} \otimes_{\mathbb{K}} V$, where $\mathbb{L} = \mathbb{K}(x_0, \underline{y}_0)$.
We consider the $\mathbb{K}[x_0, \underline{y}_0]$ derivation

$$\tilde{D} = f(x_0, \underline{y}_0)\partial_{x_0} + f_0(x_0, \underline{y}_0)\partial_{y_0} + \sum_i f_i(x_0, \underline{y}_0)\partial_{y_{i,0}}.$$

We remark that there exists an isomorphism between $\ker D \subset \mathbb{K}(x, \underline{y})$ and $\ker \tilde{D} \subset \mathbb{K}(x_0, \underline{y}_0)$.

We have then $E_D^V(x_0, \underline{y}_0) = W_{\tilde{D}}(g_1(x_0, \underline{y}_0), \dots, g_l(x_0, \underline{y}_0))$.
By Lemma 20 applied with $F = \mathbb{L}$ and $\tilde{D}$, we deduce:

$$E_D^V(x_0, \underline{y}_0) = 0 \iff g_1(x_0, \underline{y}_0), \dots, g_l(x_0, \underline{y}_0) \text{ are linearly dependent over } \ker \tilde{D}.$$

As $g_1(x_0, \underline{y}_0), \dots, g_l(x_0, \underline{y}_0)$ are linearly independent over $\mathbb{K}$, this means that $\ker \tilde{D} \neq \mathbb{K}$. Thus $\tilde{D}$ and then D has a rational first integral.

Conversely, if D has a rational first integral G_1/G_2 with degree N then we set $V = \mathbb{K}[x, \underline{y}]_{\leq N}$ and we can consider a basis $\{g_1, g_2, \dots, g_l\}$ of V where $g_1 = G_1$ and $g_2 = G_2$. We have $g_1(x_0, \underline{y}_0)/g_2(x_0, \underline{y}_0) \in \ker \tilde{D}$ and

$$g_1(x_0, \underline{y}_0) - \frac{g_1(x_0, \underline{y}_0)}{g_2(x_0, \underline{y}_0)} g_2(x_0, \underline{y}_0) = 0.$$

Thus we have a non-trivial relation over $\ker \tilde{D}$ then by Lemma 20

$$W_{\tilde{D}}(g_1(x_0,\underline{y}_0),\dots,g_l(x_0,\underline{y}_0)) = E_D^V(x_0,\underline{y}_0) = 0.$$

□

Remark 22. *The previous proof shows that if D has a rational first integral with degree N then $E_D^{\mathbb{K}[x,y]_{\leq N}}(x_0,\underline{y}_0) = 0$.*

This kind of result is not new, see [29]. We have given here a proof in order to emphasize the relation between the extactic curve and the Wronskian. The following example shows however that it is possible to have $E_D^{\mathbb{K}[x,\underline{y}]_{\leq N}}(x_0,\underline{y}_0)$ equals to zero and no rational first integral with degree N.

Example 23. *Consider the following derivation*

$$D = x\partial_x + (3x-2y)\partial_y - 3x^3\partial_{y_1}.$$

This derivation has two polynomial first integrals with degree 3:

$$P_1(x,y,y_1) = x^2y + y_1, \quad P_2(x,y,y_1) = x^3 + y_1.$$

For this derivation we have $E_D^{\mathbb{K}[x,y,y_1]_{\leq 2}}(x_0,\underline{y}_0) = 0$, but D has no rational first integral with degree 2. Indeed, a direct computation with a computer algebra system shows that the only Darboux polynomials for this derivation with degree smaller than 2 are: x, $y-x$ and their products.
Now, we explain why the second extactic curve is equal to zero. As P_1 and P_2 are first integrals we have: $\Delta P_i(x_0,\underline{y}_0;x,y(x)) = 0$, where

$$\Delta P_i(x_0,\underline{y}_0;x,\underline{y}) = P_i(x,y,y_1) - P_i(x_0,y_0,y_{1,0}).$$

Thus

$$\begin{aligned} P(x_0,\underline{y}_0;x,\underline{y}) &= x.\Delta P_1(x_0,\underline{y}_0;x,\underline{y}) - y.\Delta P_2(x_0,\underline{y}_0;x,\underline{y}) \\ &= xy_1 - yy_1 - x(x_0^2y_0 + y_{1,0}) + y(x_0^3 + y_{1,0}) \end{aligned}$$

has degree 2 in $\mathbb{K}(x_0,\underline{y}_0)[x,y,y_1]$ and satisfies

$$P(x_0,\underline{y}_0;x,y(x),y_1(x)) = 0.$$

Thus $P \in \ker \mathcal{E}_D^{\mathbb{K}[x,y,y_1]_{\leq 2}}$ and $E_D^{\mathbb{K}[x,y,y_1]_{\leq 2}}(x_0,\underline{y}_0) = 0$.

Now, we show that the computation of $\ker \mathcal{E}_D^N$ gives rational first integrals. More precisely, we are going to exhibit a structure for the elements in $\ker \mathcal{E}_D^V$.

Proposition 24. *Let V be a linear subspace of $\mathbb{K}[x,\underline{y}]$ of dimension l.*
Let $\{b_i(x,\underline{y})\}$ be a basis of V, and let $\tilde{D}$ be the following $\mathbb{K}[x_0,\underline{y}_0]$ derivation: $\tilde{D} = f(x_0,\underline{y}_0)\partial_{x_0} + f_0(x_0,\underline{y}_0)\partial_{y_0} + \sum_i f_i(x_0,\underline{y}_0)\partial_{y_{i,0}}$.
Consider $g_1(x_0,\underline{y}_0;x,\underline{y})$, ..., $g_l(x_0,\underline{y}_0;x,\underline{y})$ a basis of $\ker \mathcal{E}_D^V$ in reduced row echelon form. Then we can write each g_i in the following form:

$$g_i(x_0,\underline{y}_0;x,\underline{y}) = \sum_{j\in J_i} c_j(x_0,\underline{y}_0)b_j(x,y),$$

where J_i is a finite set and $c_j(x_0,\underline{y}_0) \in \ker \tilde{D}$.
Furthermore, for all g_i there exists j_i such that $c_{j_i}(x_0,\underline{y}_0) \notin \mathbb{K}$.

As mentioned before, there exists a natural isomorphism between the two kernels, $\ker D \subset \mathbb{K}(x,\underline{y})$ and $\ker \tilde{D} \subset \mathbb{K}(x_0,\underline{y}_0)$. Thus, this proposition says that the computation of a reduced row echelon basis of $\ker \mathcal{E}_D^V$ gives a non-trivial rational first integral: $c_{j_i}(x,\underline{y})$.
For the definition of the reduced row echelon form we can consult [25, Chapter 1].

Proof. Consider $\{g_1,\dots,g_l\}$ a basis of $\ker \mathcal{E}_D^V$ in reduced row echelon form and we set $g_i(x_0,\underline{y}_0;x,\underline{y}) = \sum_{j\in J_i} p_j(x_0,\underline{y}_0)b_j(x,y)$.

As $g_i \in \ker \mathcal{E}_D^V$, we have for all $k \leq l-1$,

$$\sum_{j\in J_i} p_j(x_0,\underline{y}_0)D^k(b_j)(x_0,\underline{y}_0) = 0.$$

We set $J_i = \{j_0, j_1 \dots, j_{k_i}\}$, then the previous equalities give:

$$\begin{pmatrix} b_{j_0}(x_0,\underline{y}_0) & \dots & b_{j_{k_i}}(x_0,\underline{y}_0) \\ D(b_{j_0})(x_0,\underline{y}_0) & \dots & D(b_{j_{k_i}})(x_0,\underline{y}_0) \\ \vdots & \dots & \vdots \\ D^{k_i}(b_{j_0})(x_0,\underline{y}_0) & \dots & D^{k_i}(b_{j_{k_i}})(x_0,\underline{y}_0) \end{pmatrix} \cdot \begin{pmatrix} p_{j_0}(x_0,\underline{y}_0) \\ p_{j_1}(x_0,\underline{y}_0) \\ \vdots \\ p_{j_{k_i}}(x_0,\underline{y}_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

As $(p_{j_0}(x_0,\underline{y}_0),\dots,p_{j_{k_i}}(x_0,\underline{y}_0)) \neq 0$, this implies $W_{\tilde{D}}(b_j(x_0,\underline{y}_0); j \in J_i) = 0$. Then, we deduce thanks to Lemma 20, that the $b_j(x_0,\underline{y}_0)$ with $j \in J_i$ are linearly related over $\ker \tilde{D}$.
Then there exists $c_j(x_0,\underline{y}_0) \in \ker \tilde{D}$ such that $\sum_{j\in J_i} c_j(x_0,\underline{y}_0)b_j(x_0,\underline{y}_0) = 0$. As $\{b_j(x_0,\underline{y}_0) \,|\, j \in J_i\}$ is a family of linearly independent elements over $\mathbb{K}$, we can suppose without loss of generality that $c_{j_0}(x_0,\underline{y}_0) = 1$ and that there exists an index j_i such that $c_{j_i}(x_0,\underline{y}_0) \notin \mathbb{K}$. Furthermore, we have:

$$\begin{aligned} 0 &= \tilde{D}\big(\sum_{j\in J_i} c_j(x_0,\underline{y}_0)b_j(x_0,\underline{y}_0)\big) \\ &= \sum_{j\in J_i} \tilde{D}(c_j)(x_0,\underline{y}_0)b_j(x_0,\underline{y}_0) + \sum_{j\in J_i} c_j(x_0,\underline{y}_0)\tilde{D}\big(b_j(x_0,\underline{y}_0)\big) \end{aligned}$$

As $c_j(x_0,\underline{y}_0) \in \ker \tilde{D}$, this implies

$$0 = \sum_{j\in J_i} c_j(x_0,\underline{y}_0)\tilde{D}\big(b_j(x_0,\underline{y}_0)\big).$$

In the same way, we get $\sum_{j\in J_i} c_j(x_0,\underline{y}_0)\tilde{D}^j\big(b_j(x_0,\underline{y}_0)\big) = 0$.
It follows that $\sum_{j\in J_i} c_j(x_0,\underline{y}_0)b_j(x,y) \in \ker \mathcal{E}_D^V$. This polynomial has the same support as the polynomial g_i and the basis $\{g_1,\dots,g_l\}$ is in reduced row echelon form, thus we get the desired result. □

Now, we are going to give an explicit statement for "if the contact between an hypersurface and an orbit is big enough then the orbit is included in the hypersurface."

Theorem 25. *Let $\underline{y}(x) = \big(y(x), y_1(x),\dots,y_n(x)\big)$ be a solution of (S_n) satisfying the initial condition $\underline{y}(x_0) = \underline{y}_0$ and let V be a linear subspace of $\mathbb{K}[x,\underline{y}]$ of dimension l.*
If $P \in \mathbb{L}\otimes_{\mathbb{K}} V$ and $P(x_0,\underline{y}_0;x,\underline{y}(x)) = 0 \mod (x-x_0)^l$ then $P(x_0,\underline{y}_0;x,\underline{y}(x)) = 0$.

Proof. Consider $P(x_0,\underline{y}_0;x,\underline{y})$ such that $P(x_0,\underline{y}_0;x,\underline{y}(x))=0 \mod (x-x_0)^l$. The Taylor expansion of P shows that $P(x_0,\underline{y}_0;x,\underline{y})\in\ker\mathcal{E}_D^V$.
We can write $P(x_0,\underline{y}_0;x,\underline{y})$ in the following form:

$$P(x_0,\underline{y}_0;x,\underline{y})=\sum_i \lambda_i(x_0,\underline{y}_0)g_i(x_0,\underline{y}_0;x,\underline{y}),$$

where the polynomials $g_i(x_0,\underline{y}_0;x,\underline{y})$ satisfy Proposition 24.

We have:

$$\begin{aligned} g_i(x,\underline{y}(x);x,\underline{y}) &= \sum_{j\in J_i} c_j(x,\underline{y}(x))b_j(x,\underline{y}) \\ &= \sum_{j\in J_i} c_j(x_0,\underline{y}_0)b_j(x,\underline{y}), \text{ because } c_j(x_0,\underline{y}_0)\in\ker\tilde{D}, \\ &= g_i(x_0,\underline{y}_0;x,\underline{y}). \end{aligned}$$

Furthermore, $g_i(x_0,\underline{y}_0;x_0,\underline{y}_0)=0$ because $g_i(x_0,\underline{y}_0;x,\underline{y})\in\ker\mathcal{E}_D^V$.
Thus $g_i(x,\underline{y}(x);x,\underline{y}(x))=0$.
As $g_i(x,\underline{y}(x);x,\underline{y})=g_i(x_0,\underline{y}_0;x,\underline{y})$ we get

$$0=g_i(x,\underline{y}(x);x,\underline{y}(x))=g_i(x_0,\underline{y}_0;x,\underline{y}(x)).$$

Then $P(x_0,\underline{y}_0;x,\underline{y}(x))=\sum_i\lambda_i(x_0,\underline{y}_0)g_i(x_0,\underline{y}_0;x,\underline{y}(x))=0$. □

This result means that for a generic point if the order of contact with a polynomial of degree N is bigger than $\dim_{\mathbb{K}}\mathbb{K}[x,\underline{y}]_{\leq N}$ then this order of contact is infinite.

4. Extactic curves

4.1. **Rational extactic curve.** In this subsection, we recall a classical result for the extactic curve in two variables. Let us note

$$\tilde{\mathcal{E}}_{D_0}^N(x_0,y_0)=\mathcal{E}_{D_0}^{\mathbb{K}[x,y]_{\leq N}},\quad \tilde{E}_{D_0}^N(x_0,y_0)=E_{D_0}^{\mathbb{K}[x,y]_{\leq N}},$$

and call a rational function $F\in\bar{\mathbb{K}}(x,y)$ *indecomposable* when it cannot be written $f\circ g$, with $f\in\overline{\mathbb{K}}(T)$, $g\in\overline{\mathbb{K}}(x,y)$ and $\deg(f)\geq 2$.

Theorem 26 (Bivariate rational extactic curve theorem)**.**
The derivation D_0 has an indecomposable rational first integral with degree N if and only if $\tilde{E}_{D_0}^N(x_0,y_0)=0$ and $\tilde{E}_{D_0}^{N-1}(x_0,y_0)\neq 0$. Moreover, this indecomposable first integral can always be assumed to have coefficients in $\mathbb{K}$.

This theorem says that the minimal degree of a rational first integral corresponds to the minimal index where the extactic curve vanishes. This theorem is not new, see e.g. [29, 14]. We have recalled it because we are going to generalize this result for the study of Darbouxian, Liouvillian and Riccati first integrals.

4.2. **Darbouxian extactic curve.** In this subsection we are going to apply the result of Section 3 to the derivation D_1. Then in the following $\big(y(x),y_1(x)\big)$ is a solution of (S_1') satisfying the initial condition $y(x_0)=y_0, y_1(x_0)=y_{1,0}$, where x_0, y_0 and $y_{1,0}$ are variables.

Now, we are going to generalize Theorem 26 to the Darbouxian case.

Definition 27. *We set*

$$V_1 := \mathbb{K}[x,y]_{\leq N} \oplus \mathbb{K}[x,y]_{\leq N} y_1^k, \quad l_1 = dim(V_1).$$

We denote by $\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$ *the linear map* $\mathcal{E}_{D_1}^{V_1}$ *after the specialization* $y_{1,0}=1$.
The N*-th* k*-Darbouxian extactic curve is defined by*

$$\tilde{E}_{D_1}^{N,k}(x_0,y_0) = \det\big(\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)\big).$$

Let us begin with two lemmas about this Darbouxian extactic curve.

Lemma 28. *We have the following equivalence:*

$$y_1^k P + Q \in \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$$
$$\Updownarrow$$
$$y_1^k P + y_{1,0}^k Q \in \ker \mathcal{E}_{D_1}^{V_1}.$$

This lemma says that evaluating $y_{1,0}=1$ in the definition of the extactic curve does not lose much information.

Proof. We denote by $\big(\psi(x),\psi_1(x)\big)$ the solution of (S_1') such that $\psi(x_0) = y_0$, $\psi_1(x_0)=1$.
We consider the transformation

$$T(y,y_1) = (y,\, y_{1,0}y_1).$$

We set

$$\big(\psi_T(x),\psi_{T,1}(x)\big) := T(\psi(x),\psi_1(x)\big) = \big(\psi(x),\, y_{1,0}\psi_1(x)\big).$$

Now, we are going to show that $\big(\psi_T(x),\psi_{T,1}(x)\big)$ is a solution of (S_1') with initial conditions $\psi_T(x_0)=y_0$, $\psi_{T,1}(x_0)=y_{1,0}$. Indeed,

$$\partial_x\psi_{T,1}(x) = y_{1,0}\partial_x\psi_1(x) = y_{1,0}\psi_1(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi(x)\big) = \psi_{T,1}(x)\partial_y\Big(\frac{B}{A}\Big)\big(x,\psi_T(x)\big).$$

This implies the equality $\big(y(x),y_1(x)\big) = \big(\psi_T(x),\psi_{T,1}(x)\big)$.

Now suppose that $y_1^kP+Q \in \ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$ then

$$\psi_1^k(x)P\big(x,\psi(x)\big) + Q\big(x,\psi(x)\big) = 0 \mod (x-x_0)^{l_1}.$$

Thus this equality multiplied by $y_{1,0}^k$ gives

$$y_{1,0}^k\psi_1^k(x)P\big(x,\psi(x)\big) + y_{1,0}^kQ\big(x,\psi(x)\big) = 0 \mod (x-x_0)^{l_1}.$$

It follows

$$\psi_{T,1}^k(x)P\big(x,\psi_T(x)\big) + y_{1,0}^kQ\big(x,\psi_T(x)\big) = 0 \mod (x-x_0)^{l_1}.$$

Therefore, as $\big(y(x),y_1(x)\big) = \big(\psi_T(x),\psi_{T,1}(x)\big)$ we deduce that $y_1^kP+y_{1,0}^kQ$ belongs to $\ker\mathcal{E}_{D_1}^{V_1}$.

The converse is straightforward. □

Lemma 29. *Consider a non trivial element* $y_1^kP+Q\in\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$*, then:*

- *If* $P=0$ *then* $Q\in\ker\tilde{\mathcal{E}}_{D_0}^{N}(x_0,y_0)$.
- *If* $Q=0$ *then* $P\in\ker\tilde{\mathcal{E}}_{D_0}^{N}(x_0,y_0)$.

- *If* $PQ \neq 0$ *and* $Q \notin \ker \tilde{\mathcal{E}}_{D_0}^N(x_0, y_0)$ *then:*

$$\left(D_0\left(\left(\frac{P}{Q}\right)^{1/k}\right) + A\left(\frac{P}{Q}\right)^{1/k}\partial_y\left(\frac{B}{A}\right)\right)\big(x, y(x)\big) = 0.$$

The first two cases are pathological ones, i.e. we compute the Darbouxian extactic curve but it appears that a rational first integral exists.

Proof. As $y_1^k P + Q \in \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0, y_0)$ we get using Lemma 28

$$y_1^k P(x, y) + y_{1,0}^k Q(x, y) \in \ker \mathcal{E}_{D_1}^{V_1}.$$

Thus

$$y_1(x)^k P(x, y(x)) + y_{1,0}^k Q(x, y(x)) = 0 \mod (x - x_0)^{l_1}.$$

By Theorem 25, we deduce that $y_1(x)^k P(x, y(x)) + y_{1,0}^k Q(x, y(x)) = 0$.
If $P = 0$ then we get $Q(x, y(x)) = 0 \mod (x - x_0)^{l_1}$, then $Q \in \ker \tilde{\mathcal{E}}_{D_0}^N(x_0, y_0)$.
If $Q = 0$ then we get $y_1(x)^k P(x, y(x)) = 0 \mod (x - x_0)^{l_1}$. We have $y_1(x) \neq 0$ as $y_1(x_0) = y_{1,0}$ and thus $P \in \ker \tilde{\mathcal{E}}_{D_0}^N(x_0, y_0)$.

Now we suppose that $PQ \neq 0$, and $Q \notin \ker \tilde{\mathcal{E}}_{D_0}^N(x_0, y_0)$, then $Q\big(x, y(x)\big) \neq 0$ and:

$$y_1(x)^k \frac{P}{Q}\big(x, y(x)\big) = -y_{1,0}^k,$$

and thus

$$y_1(x)\left(\frac{P}{Q}\right)^{1/k}\big(x, y(x)\big) = \xi y_{1,0}, \quad \text{where } \xi \text{ is a } k\text{-th root of } -1.$$

The derivation relative to x of this relation and the fact that $y_1(x)$ is a solution of (S_1') gives:

$$y_1(x)\partial_y\left(\frac{B}{A}\right)\big(x, y(x)\big)\left(\frac{P}{Q}\right)^{1/k}\big(x, y(x)\big)$$
$$+y_1(x)\left(\partial_x\left(\left(\frac{P}{Q}\right)^{1/k}\right)\big(x, y(x)\big) + \partial_y\left(\left(\frac{P}{Q}\right)^{1/k}\right)\big(x, y(x)\big)\frac{B}{A}\big(x, y(x)\big)\right) = 0.$$

As $y_1(x) \neq 0$ we get:

$$\left(A\left(\frac{P}{Q}\right)^{1/k}\partial_y\left(\frac{B}{A}\right) + D_0\left(\left(\frac{P}{Q}\right)^{1/k}\right)\right)\big(x, y(x)\big) = 0.$$

This gives the desired result. □

We now prove the main result about the Darbouxian extactic curve.

Theorem 30 (Darbouxian extactic curve theorem)**.**

(1) *If* $\tilde{E}_{D_1}^{N,k}(x_0, y_0) = 0$ *then the derivation* D_0 *has a* k*-Darbouxian first integral with degree smaller than* N *or a rational first integral with degree smaller than* $2N+2d-1$*. Moreover the defining equation of the first integral, (Rat) or (D), has coefficients in* $\mathbb{K}$*.*

(2) *If* D_0 *has a rational or a* k*-Darbouxian first integral with degree smaller than* N *then* $\tilde{E}_{D_1}^{N,k}(x_0, y_0) = 0$*.*

Suppose that D_0 has no rational first integral and that N is the smallest integer such that $\tilde{E}_{D_1}^{N,k}(x_0,y_0)=0$. Then, by the previous theorem, there exists a Darbouxian first integral with degree smaller than N. Moreover, this degree is minimal. Indeed, if there exists a Darbouxian first integral with degree $M<N$ then, by the second part the theorem, we have $\tilde{E}_{D_1}^{M,k}(x_0,y_0)=0$ and this contradicts the minimality of N. Therefore, as in the rational case, the smallest integer N satisfying $\tilde{E}_{D_1}^{N,k}(x_0,y_0)=0$ is related to a first integral with minimal degree.

Proof. First, consider a non trivial solution y_1^kP+Q in $\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$.
If $P=0$ then by Lemma 29 we get $Q\in\ker\tilde{\mathcal{E}}_{D_0}^{N}(x_0,y_0)$. Theorem 26 implies that the derivation D_0 has a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.
If $Q=0$, we deduce in the same way that D_0 has a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.

Now we suppose that $PQ\neq 0$.
If $Q\in\ker\tilde{\mathcal{E}}_{D_0}^{N}(x_0,y_0)$, then by Theorem 26, D_0 has a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.
Now, we suppose that Q does not belong to $\ker\mathcal{E}_{D_0}^{N}(x_0,y_0)$. By Lemma 29, we have

$$\Big(D_0((P/Q)^{1/k})+A(P/Q)^{1/k}\partial_y(B/A)\Big)\big(x,y(x)\big)=0.$$

We introduce the polynomial

$$G:=APQ(P/Q)^{-1/k}\Big(D_0((P/Q)^{1/k})+A(P/Q)^{1/k}\partial_y(B/A)\Big)$$

we have $G\big(x,y(x)\big)=0$.
If $G=0$ then Proposition 10 with $F=(P/Q)^{1/k}$ gives the existence of a k-Darbouxian first integral. Moreover, as $P,Q\in\mathbb{K}(x_0,y_0)[x,y]$, the equation of type (D) giving the existence of a Darbouxian first integral has coefficients in $\mathbb{K}$.
If $G\neq 0$ then G is a non-zero polynomial with degree smaller than $2N+2d-1$ such that $G\in\ker\tilde{\mathcal{E}}_{D_0}^{2N+2d-1}(x_0,y_0)$. By Theorem 26, D_0 has a rational first integral with degree smaller than $2N+2d-1$ with coefficients in $\mathbb{K}$. This concludes the first part of the proof.

Now, we suppose that D_0 has a rational or a k-Darbouxian first integral with degree smaller than N.
If D_0 has a k-Darbouxian first integral $\mathcal{F}$ with degree smaller than N then we have $\partial_y\mathcal{F}=(P/Q)^{1/k}$, with $P,Q\in\bar{\mathbb{K}}[x,y]$, and $\deg P,\deg Q\leq N$. Proposition 7 implies that y_1^kP/Q is a rational first integral of (S_1'). Thus, when we consider a solution of (S_1') with initial condition $y(x_0)=y_0$, $y_1(x_0)=1$ we get

$$y_1(x)^k\frac{P\big(x,y(x)\big)}{Q\big(x,y(x)\big)}=c,$$

where $c\in\bar{\mathbb{K}}(x_0,y_0)$. Thus $y_1^kP(x,y)-cQ(x,y)$ belongs to $\bar{\mathbb{K}}\otimes_{\mathbb{K}}\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$. Now as $A,B\in\mathbb{K}[x,y]$, the coefficients of $\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$ are in $\mathbb{K}(x_0,y_0)$, and so we deduce the existence of $\tilde{P},\tilde{Q}\in\mathbb{K}(x_0,y_0)[x,y]$ with degree N such that $y_1^k\tilde{P}-\tilde{Q}$ belong to $\ker\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$. Thus $\tilde{E}_{D_1}^{N,k}(x_0,y_0)=0$ and we get the desired conclusion.

If D_0 has a rational first integral P/Q with degree smaller than N, then we can suppose $P/Q \in \mathbb{K}(x,y)$ thanks to Theorem 26. Furthermore,

$$P(x,y)Q(x_0,y_0) - Q(x,y)P(x_0,y_0) \in \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0).$$

Thus $\tilde{E}_{D_1}^{N,k}(x_0,y_0) = 0$. □

As a corollary, we obtain the following result about the coefficient field extensions of the possible first integrals.

Corollary 31. *Suppose that D_0 has a k-Darbouxian first integral and no rational first integral. We have:*
There exists $F \in \mathbb{K}(x,y)$ with $\deg F \leq N$ such that equation (D) gives a k-Darbouxian first integral if and only if there exists $\tilde{F} \in \overline{\mathbb{K}}(x,y)$ with $\deg \tilde{F} \leq N$ such that equation (D) with $\tilde{F}$ gives a k-Darbouxian first integral.

Proof. Suppose that there exists $\tilde{F} \in \overline{\mathbb{K}}(x,y)$ with $\deg \tilde{F} = N$ such that equation (D) gives a k-Darbouxian first integral. Then by the second part of Theorem 30, we have $\tilde{E}_{D_1}^{N,k}(x_0,y_0) = 0$. Applying now the first part, we obtain either a rational first integral (forbidden by our assumption) or a k-Darbouxian first integral given by an equation (D) with a rational function with degree smaller than N and with coefficients in $\mathbb{K}$. □

4.3. **Liouvillian Extactic curve.** In this subsection we are going to apply the result of Section 3 to the derivation D_2. Then in the following $\big(y(x), y_1(x), y_2(x)\big)$ is a solution of (S_2') satisfying the initial condition $y(x_0) = y_0$, $y_1(x_0) = y_{1,0}$, $y_2(x_0) = y_{2,0}$ where x_0, y_0, $y_{1,0}$ and $y_{2,0}$ are variables.

We are going to generalize Theorem 26 to the Liouvillian case. We need thus to define the extactic curve in the Liouvillian case.

Definition 32. *We set*

$$V_2 := \mathbb{K}[x,y]_{\leq N} y_1^2 \oplus \mathbb{K}[x,y]_{\leq N} y_2 \oplus \mathbb{K}[x,y]_{\leq N} y_1, \quad l_2 = dim(V_2).$$

We denote by $\tilde{\mathcal{E}}_{D_2}^{N}(x_0,y_0)$ the linear map $\mathcal{E}_{D_2}^{V_2}$ after the specialization $y_{1,0} = 1$, $y_{2,0} = 0$.
The N-th Liouvillian extactic curve is

$$\tilde{E}_{D_2}^{N}(x_0,y_0) = \det\big(\tilde{\mathcal{E}}_{D_2}^{N}(x_0,y_0)\big).$$

As in the Darbouxian case, we begin with two lemmas in order to show that the specialization $y_{1,0} = 1$, $y_{2,0} = 0$ does not lose information.

Lemma 33. *We have the following equivalence:*

$$P(x,y)y_1^2 + Q(x,y)y_2 + R(x,y)y_1 \in \ker \tilde{\mathcal{E}}_{D_2}^{N}(x_0,y_0)$$
$$\Updownarrow$$
$$P(x,y)y_1^2 + Q(x,y)y_2 + \Big(y_{1,0}R(x,y) - \frac{y_{2,0}}{y_{1,0}}Q(x,y)\Big)y_1 \in \ker \mathcal{E}_{D_2}^{V_2}.$$

Proof. We are going to use the same strategy as the one used to prove Lemma 28. Here the transformation T is

$$T(y,y_1,y_2) = (y,\ y_{1,0}y_1,\ y_{1,0}^2 y_2 + y_{2,0}y_1).$$

We denote by $\big(\psi(x),\psi_1(x),\psi_2(x)\big)$ the solution of (S_2') such that $\psi(x_0)=y_0$, $\psi_1(x_0)=1$, $\psi_2(x_0)=0$.
We set

$$\begin{aligned}\big(\psi_T(x),\psi_{T,1}(x),\psi_{T,2}(x)\big) &:= T\big(\psi(x),\psi_1(x),\psi_2(x)\big)\\ &= \big(\psi(x),\ y_{1,0}\psi_1(x),\ y_{1,0}^2\psi_2(x)+y_{2,0}\psi_1(x)\big).\end{aligned}$$

A direct computation shows that $\big(\psi_T(x),\psi_{T,1}(x),\psi_{T,2}(x)\big)$ is a solution of (S_2') with initial conditions $\psi_T(x_0)=y_0$, $\psi_{T,1}(x_0)=y_{1,0}$, $\psi_{T,2}(x_0)=y_{2,0}$. Thus we have $\big(y(x),y_1(x),y_2(x)\big)=\big(\psi_T(x),\psi_{T,1}(x),\psi_{T,2}(x)\big)$.

Now suppose that $P(x,y)y_1^2+Q(x,y)y_2+R(x,y)y_1\in\ker\tilde{\mathcal{E}}_{D_2}^N(x_0,y_0)$ then

$$P(x,\psi(x))\psi_1^2(x)+Q(x,\psi(x))\psi_2(x)+R(x,\psi(x))\psi_1(x)=0\mod (x-x_0)^{l_2}.$$

Thus, the previous equality multiplied by $y_{1,0}^2$ gives

$$\begin{aligned}y_{1,0}^2P(x,\psi(x))\psi_1^2(x)+y_{1,0}^2Q(x,\psi(x))\psi_2(x)+y_{2,0}Q(x,\psi(x))\psi_1(x)&\\ -y_{2,0}Q(x,\psi(x))\psi_1(x)+y_{1,0}^2R(x,\psi(x))\psi_1(x) &= 0\mod (x-x_0)^{l_2}.\end{aligned}$$

This gives, thanks to the definition of ψ_T, $\psi_{T,1}$ and $\psi_{T,2}$,

$$\begin{aligned}P(x,\psi_T(x))\psi_{T,1}^2(x)+Q(x,\psi_T(x))\psi_{T,2}(x)&\\ +\Big(-\frac{y_{2,0}}{y_{1,0}}Q(x,\psi_T(x))+y_{1,0}R(x,\psi_T(x)\Big)\psi_{T,1}(x) &= 0\mod (x-x_0)^{l_2}.\end{aligned}$$

Therefore, as $\big(y(x),y_1(x),y_2(x)\big)=\big(\psi_T(x),\psi_{T,1}(x),\psi_{T,2}(x)\big)$, we get

$$P(x,y)y_1^2+Q(x,y)y_2+\Big(y_{1,0}R(x,y)-\frac{y_{2,0}}{y_{1,0}}Q(x,y)\Big)y_1\in\ker\mathcal{E}_{D_2}^{V_2}.$$

The converse is straightforward. □

Lemma 34. *Consider a non trivial element* $P(x,y)y_1^2+Q(x,y)y_2+R(x,y)y_1$ *in* $\ker\tilde{\mathcal{E}}_{D_2}^N(x_0,y_0)$, *then:*

- *If* $Q=0$ *then* $Py_1+R\in\ker\tilde{\mathcal{E}}_{D_1}^N(x_0,y_0)$.
- *If* $Q\neq 0$ *and* $Q\notin\ker\tilde{\mathcal{E}}_{D_0}^N(x_0,y_0)$ *then:*

$$\begin{aligned}0 &= y_1(x)\Big(D_0(P/Q)+A(P/Q)\partial_y(B/A)+A\partial_y^2(B/A)\Big)\big(x,y(x)\big)\\ &\quad +y_{1,0}D_0(R/Q)\big(x,y(x)\big)\end{aligned}$$

Proof. We have $P(x,y)y_1^2+Q(x,y)y_2+R(x,y)y_1\in\ker\tilde{\mathcal{E}}_{D_2}^N(x_0,y_0)$, then Lemma 33 and Theorem 25 gives, as in the proof of Lemma 29,

$$P(x,y(x))y_1(x)^2+Q(x,y(x))y_2(x)+\Big(y_{1,0}R(x,y(x))-\frac{y_{2,0}}{y_{1,0}}Q(x,y(x))\Big)y_1(x)=0.$$

If $Q=0$, then we have

$$P(x,y(x))y_1(x)+y_{1,0}R(x,y(x))=0.$$

Thus $Py_1+y_{1,0}R\in\ker\mathcal{E}_{D_1}^{V_1}$ and by Lemma 28 we get that Py_1+R belongs to $\ker\tilde{\mathcal{E}}_{D_1}^N(x_0,y_0)$.

If $Q \neq 0$ and $Q \notin \ker \tilde{\mathcal{E}}^N_{D_0}(x_0, y_0)$ then $Q\big(x, y(x)\big) \neq 0$. We set $F = P/Q$ and $G = R/Q$ then we have:

$$F\big(x, y(x)\big)y_1(x) + \frac{y_2(x)}{y_1(x)} - \frac{y_{2,0}}{y_{1,0}} + y_{1,0}G\big(x, y(x)\big) = 0.$$

The derivation relatively to x of the previous expression and the relation given by the differential system (S_2') gives:

$$\begin{aligned} 0 \; = \; & \partial_x F\big(x, y(x)\big)y_1(x) + \partial_y F\big(x, y(x)\big)\frac{B}{A}\big(x, y(x)\big)y_1(x) \\ & + F\big(x, y(x)\big)y_1(x)\partial_y\Big(\frac{B}{A}\Big)\big(x, y(x)\big) + y_1(x)\partial_y^2\Big(\frac{B}{A}\Big)\big(x, y(x)\big) \\ & + y_{1,0}\partial_x G\big(x, y(x)\big) + y_{1,0}\partial_y G\big(x, y(x)\big)\frac{B}{A}\big(x, y(x)\big). \end{aligned}$$

As $y_1(x)$ is a common factor of the first terms and $y_{1,0}$ is a common factor of the last terms we get:

$$\begin{aligned} 0 \; = \; & y_1(x)\bigg(\partial_x F\big(x, y(x)\big) + \partial_y F\big(x, y(x)\big)\frac{B}{A}\big(x, y(x)\big) + F\big(x, y(x)\big)\partial_y\Big(\frac{B}{A}\Big)\big(x, y(x)\big) \\ & + \partial_y^2\Big(\frac{B}{A}\Big)\big(x, y(x)\big)\bigg) + y_{1,0}\Big(\partial_x G\big(x, y(x)\big) + \partial_y G\big(x, y(x)\big)\frac{B}{A}\big(x, y(x)\big)\Big). \end{aligned}$$

Thus

$$0 = y_1(x)\Big(D_0(F) + (AF)\partial_y(B/A) + A\partial_y^2(B/A)\Big)\big(x, y(x)\big) + y_{1,0}D_0(G)\big(x, y(x)\big).$$

This gives the desired conclusion. □

Now we can state the generalization of Theorem 26 for the Liouvillian case.

Theorem 35 (Liouvillian extactic curve Theorem)**.**

(1) *If $\tilde{E}^N_{D_2}(x_0, y_0) = 0$ then the derivation D_0 has a Liouvillian first integral with degree smaller than N or a Darbouxian first integral with degree smaller than $2N+3d-1$ or a rational first integral with degree smaller than $4N + 8d - 3$. Moreover the defining equation of the first integral, equation (Rat), (D) or (L), has coefficients in $\mathbb{K}$.*

(2) *If D_0 has a rational or a Darbouxian or a Liouvillian first integral with degree smaller than N then $\tilde{E}^N_{D_2}(x_0, y_0) = 0$.*

Proof. First suppose that $\tilde{E}^N_{D_2}(x_0, y_0) = 0$, and consider a non trivial element

$$P(x, y)y_1^2 + Q(x, y)y_2 + R(x, y)y_1 \in \ker \tilde{\mathcal{E}}^N_{D_2}(x_0, y_0).$$

If $Q = 0$ then by Lemma 34, $Py_1 + R \in \ker \tilde{\mathcal{E}}^N_{D_1}(x_0, y_0)$ and Theorem 30 gives the existence of a Darbouxian first integral with degree smaller than N and defining equation (D) with coefficients in $\mathbb{K}$ or a first integral with degree smaller than

$2N+2d-1$ with coefficients in $\mathbb{K}$.

If $Q \neq 0$ and $Q \in \ker \tilde{\mathcal{E}}_{D_0}^N(x_0,y_0)$ then by Theorem 26 there exists a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.

If $Q \neq 0$ and $Q \notin \ker \tilde{\mathcal{E}}_{D_0}^N(x_0,y_0)$ then we have two situations:
In the first situation we have:

$$D_0(P/Q) + A(P/Q)\partial_y(B/A) + A\partial_y^2(B/A) = 0.$$

In this case, Proposition 10 gives the existence of a Liouvillian first integral with degree smaller than N, since $\deg(P), \deg(Q) \leq N$. Moreover, the equation of type (L) giving the existence of a Liouvillian first integral has coefficients in $\mathbb{K}$ since $P, Q \in \mathbb{K}(x_0,y_0)[x,y]$.

In the second situation we have

$$D_0(P/Q) + A(P/Q)\partial_y(B/A) + A\partial_y^2(B/A) \neq 0$$

and thanks to Lemma 34

$$\begin{aligned} &y_1(x)\Big(D_0(P/Q) + A(P/Q)\partial_y(B/A) + A\partial_y^2(B/A)\Big)\big(x,y(x)\big) \\ &+y_{1,0}D_0(R/Q)\big(x,y(x)\big) = 0. \end{aligned}$$

In this case, we set $P_1 = A^2Q^2\big(D_0(P/Q) + A(P/Q)\partial_y(B/A) + A\partial_y^2(B/A)\big)$, $Q_1 = A^2Q^2D_0(R/Q)$, and P_1, Q_1 are polynomials with degree smaller than $2N+3d-1$. The previous equality gives $y_1P_1 + y_{1,0}Q_1 \in \ker \mathcal{E}_{D_1}^{2N+3d-1,1}$. Thus thanks to Lemma 28 we get $y_1P_1 + Q_1 \in \ker \tilde{\mathcal{E}}_{D_1}^{2N+3d-1,1}(x_0,y_0)$. Theorem 30 gives the existence of a Darbouxian first integral with degree smaller than $2N+3d-1$ or the existence of a rational first integral with degree smaller than $4N+8d-3$ with coefficients in $\mathbb{K}$.

The second part of the theorem is straightforward and is proved with the strategy used in Theorem 30. □

As before, we deduce that the computation of $F \in \mathbb{K}(x,y)$ is not restrictive.

Corollary 36. *Suppose that D_0 has a Liouvillian first integral and no Darbouxian nor rational first integral. There exists $F \in \mathbb{K}(x,y)$ with $\deg F \leq N$ such that equation (L) gives a Liouvillian first integral if and only if there exists $\tilde{F} \in \overline{\mathbb{K}}(x,y)$ with $\deg \tilde{F} \leq N$ such that equation (L) with $\tilde{F}$ gives a Liouvillian first integral.*

4.4. **Riccati extactic curve.** In this subsection we are going to apply the result of Section 3 to the derivation D_3. Then in the following $\big(y(x), y_1(x), y_2(x), y_3(x)\big)$ is a solution of (S_3') satisfying the initial condition $y(x_0) = y_0$, $y_1(x_0) = y_{1,0}$, $y_2(x_0) = y_{2,0}, y_3(x_0) = y_{3,0}$ where x_0, y_0, $y_{1,0}$, $y_{2,0}$ and $y_{3,0}$ are variables.

Now, we are going to generalize Theorem 26 to the Riccati case and we follow the same strategy as before.

Definition 37. *We set*

$$V_3 := \mathbb{K}[x,y]_{\leq N}y_1^4 \oplus \mathbb{K}[x,y]_{\leq N}(3y_2^2 - 2y_3y_1) \oplus \mathbb{K}[x,y]_{\leq N}y_1^2, \quad l_3 = dim(V_3).$$

We denote by $\tilde{\mathcal{E}}_{D_3}^{N}(x_0,y_0)$ *the linear map* $\mathcal{E}_{D_3}^{V_3}$ *after the specialization* $y_{1,0}=1$, $y_{2,0}=0$, $y_{3,0}=0$.
The N*-th Riccati extactic curve is defined by*

$$\tilde{E}_{D_3}^{N}(x_0,y_0)=\det\big(\tilde{\mathcal{E}}_{D_3}^{N}(x_0,y_0)\big).$$

As before, we begin by proving two Lemmas.

Lemma 38. *We have the following equivalence:*

$$4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)+R(x,y)y_1^2\in\ker\tilde{\mathcal{E}}_{D_3}^{N}(x_0,y_0)$$
$$\Updownarrow$$
$$\begin{aligned}&4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)\\&+\left(R(x,y)y_{1,0}^2-\left(3\frac{y_{2,0}^2}{y_{1,0}^2}-2\frac{y_{3,0}}{y_{1,0}}\right)Q(x,y)\right)y_1^2\quad\in\ker\mathcal{E}_{D_3}^{V_3}\end{aligned}$$

Proof. With the same strategy as the one used to prove Lemma 28, we are going to obtain the desired equivalence.

We denote by $\psi(x)$ the solution of (S_3') such that $\psi(x_0)=y_0$, $\psi_1(x_0)=1$, $\psi_2(x_0)=0$, $\psi_3(x_0)=0$.
We consider the transformation

$$T(y,y_1,y_2,y_3)=(y,\ y_{1,0}y_1,\ y_{1,0}^2y_2+y_{2,0}y_1,\ y_{1,0}^3y_3+3y_{1,0}y_{2,0}y_2+y_{3,0}y_1).$$

We set

$$\begin{aligned}&\big(\psi_T(x),\psi_{T,1}(x),\psi_{T,2}(x),\psi_{T,3}(x)\big):=T(\psi(x),\psi_1(x),\psi_2(x),\psi_3(x))\\=\ &\big(\psi(x),\ y_{1,0}\psi_1(x),\ y_{1,0}^2\psi_2(x)+y_{2,0}\psi_1(x),\ y_{1,0}^3\psi_3(x)+3y_{1,0}y_{2,0}\psi_2(x)+y_{3,0}\psi_1(x)\big).\end{aligned}$$

A direct computation shows that $\big(\psi_T(x),\psi_{T,1}(x),\psi_{T,2}(x),\psi_{T,3}(x)\big)$ is a solution of (S_3') with initial conditions

$$\psi_T(x_0)=y_0,\psi_{T,1}(x_0)=y_{1,0},\psi_{T,2}(x_0)=y_{2,0},\psi_{T,3}(x_0)=y_{3,0}.$$

Then we have the equality $\big(y(x),y_1(x),y_2(x),y_3(x)\big)=\big(\psi_T(x),\psi_{T,1}(x),\psi_{T,2}(x),\psi_{T,3}(x)\big)$.

Now suppose that $4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)+R(x,y)y_1^2\in\ker\tilde{\mathcal{E}}_{D_3}^{N}(x_0,y_0)$ then

$$4P(x,\psi(x))\psi_1^4(x)+Q(x,\psi(x))(3\psi_2^2(x)-2\psi_3(x)\psi_1(x))+R(x,\psi(x))\psi_1^2(x)=0\mod(x-x_0)^{l_3}.$$

We multiply by $y_{1,0}^4$ the previous equality. Then, in order to bring $\psi_{T,3}$ out, we use the following equality:

$$y_0^4\big(3\psi_2^2-2\psi_3\psi_1\big)=3y_0^4\psi_2^2-2\big(y_{1,0}^3\psi_3+3y_{1,0}y_{2,0}\psi_2+y_{3,0}\psi_1\big)y_{1,0}\psi_1+6y_{1,0}^2y_{2,0}\psi_2\psi_1+2y_{3,0}y_{1,0}\psi_1^2.$$

This gives:

$$\begin{aligned}&4P(x,\psi(x))y_{1,0}^4\psi_1^4(x)\\&+Q(x,\psi(x))\Big(3y_{1,0}^4\psi_2^2(x)-2\big(y_{1,0}^3\psi_3(x)+3y_{1,0}y_{2,0}\psi_2(x)+y_{3,0}\psi_1(x)\big)y_{1,0}\psi_1(x)\\&+6y_{1,0}^2y_{2,0}\psi_2(x)\psi_1(x)+2y_{3,0}y_{1,0}\psi_1^2(x)\Big)+R(x,\psi(x))y_{1,0}^4\psi_1^2(x)\\&=0\mod(x-x_0)^{l_3}.\end{aligned}$$

By definition of ψ_1 and ψ_3, we get:

$$
\begin{aligned}
&4P(x,\psi(x))\psi_{T,1}^4(x)\\
&+Q(x,\psi(x))\Big(3y_{1,0}^4\psi_2^2(x)-2\psi_{T,3}(x)\psi_{T,1}(x)+6y_{1,0}^2y_{2,0}\psi_2(x)\psi_1(x)+2y_{3,0}y_{1,0}\psi_1^2(x)\Big)\\
&+R(x,\psi(x))y_{1,0}^2\psi_{T,1}^2(x)=0 \mod (x-x_0)^{l_3}.
\end{aligned}
$$

Now, in order to bring $\psi_{T,2}$ out, we use the equality

$$3y_{1,0}^4\psi_2^2=3(y_{1,0}^2\psi_2+y_{2,0}\psi_1)^2-6y_{1,0}^2y_{2,0}\psi_1\psi_2-3y_{2,0}^2\psi_1^2.$$

This gives:

$$
\begin{aligned}
&4P(x,\psi(x))\psi_{T,1}^4(x)\\
&+Q(x,\psi(x))\Big(3(y_{1,0}^2\psi_2(x)+y_{2,0}\psi_1(x))^2-6y_{1,0}^2y_{2,0}\psi_2(x)\psi_1(x)-3y_{2,0}^2\psi_1^2(x)\\
&-2\psi_{T,3}(x)\psi_{T,1}(x)+6y_{1,0}^2y_{2,0}\psi_2(x)\psi_1(x)+2y_{3,0}y_{1,0}\psi_1^2(x)\Big)\\
&+R(x,\psi(x))y_{1,0}^2\psi_{T,1}^2(x)=0 \mod (x-x_0)^{l_3}.
\end{aligned}
$$

We can simplify by $6y_{1,0}^2y_{2,0}\psi_2\psi_1$ the previous equality. Then by definition of $\psi_{T,2}$ we obtain:

$$
\begin{aligned}
&4P(x,\psi(x))\psi_{T,1}^4(x)\\
&+Q(x,\psi(x))\Big(3\psi_{T,2}^2(x)-3y_{2,0}^2\psi_1^2(x)-2\psi_{T,3}(x)\psi_{T,1}(x)+2y_{3,0}y_{1,0}\psi_1^2(x)\Big)\\
&+R(x,\psi(x))y_{1,0}^2\psi_{T,1}^2(x)=0 \mod (x-x_0)^{l_3}.
\end{aligned}
$$

At last, we factorize by ψ_1^2, and we get:

$$
\begin{aligned}
&4P(x,\psi_T(x))\psi_{T,1}^4(x)\\
&+Q(x,\psi_T(x))(3\psi_{T,2}(x)^2-2\psi_{T,3}(x)\psi_{T,1}(x))\\
&\left(\left(-3\tfrac{y_{2,0}^2}{y_{1,0}^2}+2\tfrac{y_{3,0}}{y_{1,0}}\right)Q(x,\psi_T(x))+R(x,\psi_T(x))y_{1,0}^2\right)\psi_{T,1}^2(x)=0 \mod (x-x_0)^{l_3}.
\end{aligned}
$$

Therefore, as $\big(y(x),y_1(x),y_2(x),y_3(x)\big)=\big(\psi_T(x),\psi_{T,1}(x),\psi_{T,2}(x),\psi_{T,3}(x)\big)$ we deduce

$$4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)+\left(R(x,y)y_{1,0}^2-\left(3\frac{y_{2,0}^2}{y_{1,0}^2}-2\frac{y_{3,0}}{y_{1,0}}\right)Q(x,y)\right)y_1^2$$

belongs to $\ker\mathcal{E}_{D_3}^{V_3}$.

The converse is straightforward. □

Lemma 39. *Consider a non trivial element*

$$4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)+R(x,y)y_1^2\in\ker\tilde{\mathcal{E}}_{D_3}^N(x_0,y_0),$$

then:

- *If* $Q=0$ *then* $4Py_1^2+R\in\ker\tilde{\mathcal{E}}_{D_1}^{N,2}(x_0,y_0)$.
- *If* $Q\neq 0$ *and* $Q\notin\ker\tilde{\mathcal{E}}_{D_0}^N(x_0,y_0)$ *then:*

$$
\begin{aligned}
0 \;=\;& y_1(x)^2\Big(4D_0(P/Q)+8A(P/Q)\partial_y(B/A)-2A\partial_y^3(B/A)\Big)\big(x,y(x)\big)\\
&+y_{1,0}^2D_0(R/Q)\big(x,y(x)\big).
\end{aligned}
$$

Proof. We have $4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)+R(x,y)y_1^2 \in \ker \tilde{\mathcal{E}}_{D_3}^N(x_0,y_0)$, then Lemma 38 and Theorem 25 gives:

$$(\star)\qquad 4P\big(x,y(x)\big)y_1^4(x)+Q\big(x,y(x)\big)(3y_2^2(x)-2y_3(x)y_1(x)) \\ +\left(R\big(x,y(x)\big)y_{1,0}^2-\left(3\frac{y_{2,0}^2}{y_{1,0}^2}-2\frac{y_{3,0}}{y_{1,0}}\right)Q\big(x,y(x)\big)\right)y_1^2(x)=0.$$

If $Q=0$ then we have $4P\big(x,y(x)\big)y_1^4(x)+y_{1,0}^2R\big(x,y(x)\big)y_1^2(x)=0$. As $y_1(x)\neq 0$, we get $4P\big(x,y(x)\big)y_1^2(x)+y_{1,0}^2R\big(x,y(x)\big)=0$. And thus $4Py_1^2+R$ belongs to $\ker \tilde{\mathcal{E}}_{D_1}^{N,2}(x_0,y_0)$ by Lemma 28.

If $Q\neq 0$ and $Q\notin \ker \tilde{\mathcal{E}}_{D_0}^N(x_0,y_0)$ then $Q\big(x,y(x)\big)\neq 0$. We set $F=P/Q$ and $G=R/Q$ then by dividing $(\star)$ by $Q\big(x,y(x)\big)y_1^2(x)$, we have:

$$(\star\star)\quad 4F\big(x,y(x)\big)y_1^2(x)+\left(3\frac{y_2^2(x)}{y_1^2(x)}-2\frac{y_3(x)}{y_1(x)}\right)+y_{1,0}^2G\big(x,y(x)\big)-\left(3\frac{y_{2,0}^2}{y_{1,0}^2}-2\frac{y_{3,0}}{y_{1,0}}\right)=0.$$

The derivation relatively to x of $3\frac{y_2^2(x)}{y_1^2(x)}-2\frac{y_3(x)}{y_1(x)}$ and the relation given by the differential system (S_3') gives:

$$\begin{aligned}\partial_x\left(3\frac{y_2^2(x)}{y_1^2(x)}-2\frac{y_3(x)}{y_1(x)}\right) &= \Bigg[3\left(2y_2\Big[y_2\partial_y\Big(\frac{B}{A}\Big)+y_1^2\partial_y^2\Big(\frac{B}{A}\Big)\Big]\frac{1}{y_1^2}\right)\\ &\quad -6y_2^2y_1\partial_y\Big(\frac{B}{A}\Big)\frac{1}{y_1^3}\\ &\quad -2\left(y_3\partial_y\Big(\frac{B}{A}\Big)+3y_2y_1\partial_y^2\Big(\frac{B}{A}\Big)+y_1^3\partial_y^3\Big(\frac{B}{A}\Big)\right)\frac{1}{y_1}\\ &\quad +2y_3y_1\partial_y\Big(\frac{B}{A}\Big)\frac{1}{y_1^2}\Bigg](x,y(x),y_1(x),y_2(x),y_3(x))\\ &= -2y_1^2(x)\partial_y^3\Big(\frac{B}{A}\Big)\big(x,y(x)\big).\end{aligned}$$

Then the derivation relatively to x of $(\star\star)$ gives:

$$0=4A^{-1}(x,y(x))D_0(F)(x,y(x))y_1^2(x)+8F(x,y(x))y_1^2(x)\partial_y\Big(\frac{B}{A}\Big) \\ -2y_1^2(x)\partial_y^3\Big(\frac{B}{A}\Big)\big(x,y(x)\big)+y_{1,0}^2A^{-1}(x,y(x))D_0(G)(x,y(x))$$

Thus

$$\begin{aligned}0 &= y_1(x)^2\Big(4D_0(F)+8AF\partial_y(B/A)-2A\partial_y^3(B/A)\Big)\big(x,y(x)\big)\\ &\quad +y_{1,0}^2D_0(G)\big(x,y(x)\big).\end{aligned}$$

This gives the desired conclusion. □

Now we can state the generalization of Theorem 35 for the Riccati case.

Theorem 40 (Riccati extactic curve Theorem)**.**

(1) *If* $\tilde{E}^N_{D_3}(x_0,y_0)=0$ *then the derivation* D_0 *has a Riccati first integral with degree smaller than* N *or a 2-Darbouxian first integral with degree smaller than* $2N+4d-1$ *or a rational first integral with degree smaller than* $4N+10d-3$*. Moreover the defining equation of the first integral, (Rat), (D) or (Ric), has coefficients in* $\mathbb{K}$*.*
(2) *If* D_0 *has a rational or a 2-Darbouxian or a Riccati first integral with degree smaller than* N *then* $\tilde{E}^N_{D_3}(x_0,y_0)=0$*.*

Proof. First consider a non trivial solution

$$4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)+R(x,y)y_1^2\in\ker\tilde{\mathcal{E}}^N_{D_3}(x_0,y_0).$$

If $Q=0$ then by Lemma 39, then $4Py_1^2+R\in\ker\tilde{\mathcal{E}}^{N,2}_{D_1}(x_0,y_0)$ and Theorem 30 gives the existence of a 2-Darbouxian first integral with degree smaller than N and defining equation (D) with coefficients in $\mathbb{K}$ or a first integral with degree smaller than $2N+2d-1$ and with coefficients in $\mathbb{K}$.

If $Q\neq 0$ and $Q\in\ker\tilde{\mathcal{E}}^N_{D_0}(x_0,y_0)$ then by Theorem 26 there exists a rational first integral with degree smaller than N with coefficients in $\mathbb{K}$.

If $Q\neq 0$ and $Q\notin\ker\tilde{\mathcal{E}}^N_{D_0}(x_0,y_0)$ then we have two situations:
In the first situation we have:

$$4D_0(P/Q)+8A(P/Q)\partial_y(B/A)-2A\partial_y^3(B/A)=0.$$

In this case, Proposition 10 gives the existence of a Riccati first integral. As $\deg P,\deg Q\le N$, we deduce the existence of a Riccati first integral with degree smaller than N. Moreover, the equation of type (Ric) giving the existence of a Riccati first integral has coefficients in $\mathbb{K}$ since $P,Q\in\mathbb{K}(x_0,y_0)[x,y]$.
In the second situation we have

$$4D_0(P/Q)+8A(P/Q)\partial_y(B/A)-2A\partial_y^3(B/A)\neq 0$$

and thanks to Lemma 39

$$\begin{aligned}0 \;=\;& y_1(x)^2\Big(4D_0(P/Q)+8A(P/Q)\partial_y(B/A)-2A\partial_y^3(B/A)\big)\big(x,y(x)\Big)\\ &+y_{1,0}^2D_0(R/Q)\big(x,y(x)\big).\end{aligned}$$

In this case, we set

$$P_1=A^3Q^2\big(4D_0(P/Q)+8A(P/Q)\partial_y(B/A)-2A\partial_y^3(B/A)\big),$$

$$Q_1=A^3Q^2D_0(R/Q)$$

and we obtain $y_1^2P_1+y_{1,0}^2Q_1\in\ker\tilde{\mathcal{E}}^{2N+4d-1,2}_{D_1}(x_0,y_0,y_{1,0})$. Therefore by Lemma 28, $y_1^2P_1+Q_1\in\ker\mathcal{E}^{2N+4d-1,2}_{D_1}(x_0,y_0)$, thus Theorem 30 gives the existence of a 2-Darbouxian first integral with degree smaller than $2N+4d-1$ or the existence of a rational first integral with degree smaller than $4N+10d-3$. Moreover the equation giving this first integral has coefficients in $\mathbb{K}$ since $P_1,Q_1\in\mathbb{K}(x_0,y_0)[x,y]$.

The second part of the theorem is proved with the strategy used in Theorem 30. □

As before, we remark that the computation of $F\in\mathbb{K}(x,y)$ is not restrictive.

Corollary 41. *Suppose that D_0 has a Riccati first integral and no 2-Darbouxian nor rational first integral. We have:*
There exists $F \in \mathbb{K}(x,y)$ with $\deg F \leq N$ such that equation (Ric) gives a Riccati first integral if and only if there exists $\tilde{F} \in \overline{\mathbb{K}}(x,y)$ with $\deg \tilde{F} \leq N$ such that equation (Ric) gives a Riccati first integral.

5. Evaluations of extactic curves

In our algorithms we will not compute the extactic curves as polynomials in x_0, y_0. We will only compute the extactic curves evaluated at a random point $(x_0^\star, y_0^\star) \in \mathbb{K}^2$. If the extactic curve is a non-zero polynomial, then almost surely its evaluation at a random point will not be zero. However, theoretically this can happen, and thus we want to bound the algebraic set on which such kind of bad situations can happen.

Definition 42. *We denote by $\Sigma_{D_r,N,k}$ (where k is omitted when $r \neq 1$) the following algebraic variety:*

$$\Sigma_{D_r,N,k} = \mathcal{V}\big(p \times p \text{ minors of } \tilde{\mathcal{E}}_{D_r}^{N,k}(x_0,y_0), \text{ where } p = \text{rank } \tilde{\mathcal{E}}_{D_r}^{N,k}\big).$$

Lemma 43. *Let $(x_0^\star, y_0^\star) \in \mathbb{K}^2$, $1 \leq r \leq 3$ and $P \in \ker \tilde{\mathcal{E}}_{D_r}^{N,k}(x_0^\star, y_0^\star)$.*
We consider $y_\star(x), \dots, y_{3,\star}(x)$ a solution of (S_3') with initial condition $y_\star(x_0^\star) = y_0^\star$, $y_{1,\star}(x_0^\star) = 1$, $y_{2,\star}(x_0^\star) = y_{3,\star}(x_0^\star) = 0$.
If $(x_0^\star, y_0^\star) \notin \Sigma_{D_r,N,k}$ then $P\big(x, y_\star(x), \dots, y_{r,\star}(x)\big) = 0$.

Proof. By Lemma 28, Lemma 33 and Lemma 38, we have

$$\dim_{\mathbb{K}(x_0,y_0)} \ker \tilde{\mathcal{E}}_{D_r}^{N,k}(x_0,y_0) = \dim_{\mathbb{L}_r} \ker \mathcal{E}_{D_r}^{V_r},$$

where $\mathbb{L}_r = \mathbb{K}(x_0, y_0, y_{1,0}, \dots, y_{r,0})$.
If $(x_0^\star, y_0^\star) \notin \Sigma_{D_r,N,k}$ then

$$\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_r}^{N,k}(x_0^\star, y_0^\star) = \dim_{\mathbb{K}(x_0,y_0)} \ker \tilde{\mathcal{E}}_{D_r}^{N,k}(x_0,y_0) = \dim_{\mathbb{L}_r} \ker \mathcal{E}_{D_r}^{V_r}.$$

Thus in this situation, if $P \in \ker \tilde{\mathcal{E}}_{D_r}^{N,k}(x_0^\star, y_0^\star)$ then there exists an element $\mathcal{P}(x_0, y_0, \dots, y_{r,0}; x, y, \dots, y_r)$ in $\ker \mathcal{E}_{D_r}^{V_r}$ such that

$$\mathcal{P}(x_0^\star, y_0^\star, 1, 0, 0; x, y, y_1, \dots, y_r) = P(x, y, \dots, y_r).$$

By Theorem 25, we have

$$\mathcal{P}(x_0, y_0, \dots, y_{r,0}; x, y(x), \dots, y_r(x)) = 0$$

then after the specialization we get

$$P\big(x, y_\star(x), \dots, y_{r,\star}(x)\big) = 0.$$

□

In the following, we will need some explicit bounds on the degree of the minors of $\tilde{\mathcal{E}}_{D_r}^{N,k}(x_0,y_0)$.

Lemma 44. • *The degree of a minor of $\tilde{\mathcal{E}}_{D_0}^{N}(x_0,y_0)$ is smaller than*

$$\mathcal{B}_0(d,N) := \frac{N(N+1)(N+2)}{2} + (d-1)\frac{(N+1)^2(N+2)^2 - (N+1)(N+2)}{8}.$$

- *The degree of a minor of* $\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$ *is smaller than*

$$\begin{aligned}\mathcal{B}_1(d,N) &:= Nl_1 + \frac{(2d-1)(l_1-1)l_1}{2}\\ &= N(N+1)(N+2) + (2d-1)\frac{(N+1)^2(N+2)^2-(N+1)(N+2)}{2},\end{aligned}$$

where $l_1 = \dim_{\mathbb{K}} V_1$*, see Definition 27.*

- *The degree of a minor of* $\tilde{\mathcal{E}}_{D_2}^{N}(x_0,y_0)$ *is smaller than*

$$\begin{aligned}\mathcal{B}_2(d,N) &:= Nl_2 + \frac{(3d-1)(l_2-1)l_2}{2}\\ &= \frac{3N(N+1)(N+2)}{2} + \frac{3d-1}{2}\Big[\Big(\frac{3}{2}(N+1)(N+2)\Big)^2 - \frac{3}{2}(N+1)(N+2)\Big],\end{aligned}$$

where $l_2 = \dim_{\mathbb{K}} V_2$*, see Definition 32.*

- *The degree of a minor of* $\tilde{\mathcal{E}}_{D_3}^{N}(x_0,y_0)$ *is smaller than*

$$\begin{aligned}\mathcal{B}_3(d,N) &:= Nl_2 + \frac{(4d-1)(l_3-1)l_3}{2}\\ &= \frac{3N(N+1)(N+2)}{2} + \frac{4d-1}{2}\Big[\Big(\frac{3}{2}(N+1)(N+2)\Big)^2 - \frac{3}{2}(N+1)(N+2)\Big],\end{aligned}$$

where $l_3 = \dim_{\mathbb{K}} V_3$*, see Definition 37.*

Proof. The degree of a minor of $\tilde{\mathcal{E}}_{D_0}^{N}(x_0,y_0)$ is smaller than the degree of the extactic curve. The announced bound is given in [10]. We apply here the same strategy for $\tilde{\mathcal{E}}_{D_1}^{N,k}(x_0,y_0)$:
Let $\{v_i\}$ be basis of V_1. The degree in x_0, y_0 of a minor $\mathcal{M}$ of $\tilde{\mathcal{E}}_{D_1}^{N}(x_0,y_0)$ satisfies

$$\deg(\mathcal{M}) \leq \sum_{j=0}^{l_1-1} \deg D_1^j(v_i).$$

As $\deg\big(D_1^j(v_i)\big) \leq j(2d-1)+N$, we get

$$\deg(\mathcal{M}) \leq \sum_{j=0}^{l_1-1} j(2d-1)+N \leq Nl_1 + (2d-1)\sum_{j=0}^{l_1-1} j \leq Nl_1 + \frac{(2d-1)(l_1-1)l_1}{2}.$$

The bounds for $\tilde{\mathcal{E}}_{D_2}^{N}(x_0,y_0), \tilde{\mathcal{E}}_{D_3}^{N}(x_0,y_0)$ are obtained in the same way. □

Corollary 45. *The algebraic variety* $\Sigma_{D_r,N,k}$ *is included in an algebraic hypersurface with degree smaller than* $\mathcal{B}_r(d,N)$*.*

The following set will be also useful to characterize some special situations.

Definition 46. *We denote by* $\mathfrak{S}_N$ *the following set:*
If D_0 *has no rational first integrals of degree* $\leq N$ *then*

$$\mathfrak{S}_N = \left\{ (x_0,y_0) \in \mathbb{K}^2 \;\middle|\; \begin{array}{l} (x_0,y_0) \text{ vanishes an irreducible} \\ \text{Darboux polynomial of degree } \leq N \end{array} \right\}.$$

If D *has an indecomposable rational first integral* P/Q *of degree* $p \leq N$ *then*

$$\mathfrak{S}_N = \left\{ (x_0,y_0) \in \mathbb{K}^2 \;\middle|\; \begin{array}{l} (x_0,y_0) \textit{ vanishes an irreducible} \\ \textit{Darboux polynomial of degree} < p \end{array} \right\}.$$

This set corresponds to non-generic situations. Thus we will try to avoid them. Now, we give a bound on this set:

Lemma 47. *The algebraic variety $\mathfrak{S}_N$ is included in an algebraic hypersurface with degree smaller than $\big(\frac{d(d+1)}{2}+5\big)N$.*

Proof. If D_0 has no rational first integral then by the Darboux-Jouanolou theorem D_0 has at most $d(d+1)/2$ irreducible Darboux polynomials, see e.g. [24] or [36, 18]. Therefore, if $(x_0,y_0) \in \mathfrak{S}_N$ then (x_0,y_0) vanishes the product of $d(d+1)/2$ bivariate polynomials with degree smaller than N. This gives a bound on the degree which is lower than the bound of the Lemma.

If D_0 has a rational first integral with degree $p \leq N$ then all irreducible Darboux polynomials divide a linear combination $\lambda P - \mu Q$ where P/Q is an indecomposable rational first integral with degree p. By the Darboux-Jouanolou theorem we know that all but finitely many irreducible Darboux polynomials are of the form $\lambda P - \mu Q$ and have degree p. The set $\sigma(P,Q)$ of $(\lambda:\mu) \in \mathbb{P}^1(\overline{\mathbb{K}})$ such that $\lambda P - \mu Q$ is reducible or has a degree strictly smaller than p is the set of remarkable values. Sometimes this set is called the spectrum of P/Q. It is proved in [12] that $|\sigma(P,Q)| \leq d(d+1)/2+5$. So if $(x_0,y_0) \in \mathfrak{S}_N$ then it vanishes a polynomial $\lambda P - \mu Q$ where $(\lambda:\mu)$ belongs to $\sigma(P,Q)$. Therefore, if D_0 has a rational first integral then $\mathfrak{S}_N$ is included in an algebraic hypersurface with degree smaller than $\big(\frac{d(d+1)}{2}+5\big)N$. □

6. First integral algorithms

In the following sections we are going to describe our algorithms. As mentionned before we are going to compute rational first integrals for the derivations D_0, D_1, D_2, D_3. These rational first integrals are computed thanks to the extactic curves. We are going to consider one point $(x_0^\star, y_0^\star) \in \mathbb{K}^2$ and to compute a non trivial element in $\ker \tilde{\mathcal{E}}_{D_r}^N(x_0^\star, y_0^\star)$. We will see that if $(x_0^\star, y_0^\star)$ avoids an algebraic variety then we can compute a symbolic first integral from this element.

Compute flow series

`Input:` $A(x,y), B(x,y) \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$, $r \in [[0;3]]$.
`Output:` $r+1$ series $y_\star(x), \dots, y_{r,\star}(x)$ solutions of $(S'_r) \mod (x-x_0^\star)^\sigma$ where $\sigma = \min(r+1,3)\frac{(N+1)(N+2)}{2}$, with initial condition $y_\star(x_0^\star) = y_0^\star$, $y_{1,\star}(x_0^\star) = 1$, $y_{2,\star}(x_0^\star) = y_{3,\star}(x_0^\star) = 0$.

This subroutine is performed with the algorithm given in [6].

For the following subroutine, we need a weighted degree in order to specified the output. We use the following weighted degree:
$\text{w-deg}\big(P(x,y,y_1,y_2,y_3)\big) = \deg\Big(P\big(x,y,y_1^{N+1},y_2^{2N+2},y_3^{3N+3}\big)\Big)$.

This weighted degree will be useful for the following reason:

$$\text{w-deg}\big(y_1^k P(x,y) + Q(x,y)\big) > \text{w-deg}\big(\mathcal{Q}(x,y)\big),$$

for all $P, Q, \mathcal{Q} \in \mathbb{K}[x,y]_{\leq N}$, when $P \neq 0$.

Now, if we have an element $y_1 P(x,y) + Q(x,y) \in \ker \tilde{\mathcal{E}}_{D_1}^N(x_0^\star, y_0^\star)$ then we can build a Darbouxian first integral from P and Q. However, if we have obtained an element $\mathcal{Q}(x,y) \in \ker \tilde{\mathcal{E}}_{D_1}^N(x_0^\star, y_0^\star)$ then we can build a rational first integral. Indeed, thanks to Lemma 29 we know that $\mathcal{Q}(x,y) \in \ker \tilde{\mathcal{E}}_{D_0}^N(x_0^\star, y_0^\star)$.
Therefore, the computation of an element in $\ker \tilde{\mathcal{E}}_{D_1}^N(x_0^\star, y_0^\star)$ with minimal weighted degree allows one to get a rational first integral instead of a Darbouxian first integral. This means that the computation of a non trivial in $\ker \tilde{\mathcal{E}}_{D_r}^N(x_0^\star, y_0^\star)$ with minimal weighted degree gives symbolic first integral with degree bounded by N in the simplest class of first integral.

Compute solution extactic kernel
`Input:` $A(x,y), B(x,y) \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$, $r \in [[0;3]]$,
$r+1$ series $y_\star(x), \dots, y_{r,\star}(x)$ solutions of $(S'_r) \mod (x - x_0^\star)^\sigma$
where $\sigma = \min(r+1, 3)\frac{(N+1)(N+2)}{2}$, with initial condition $y_\star(x_0^\star) = y_0^\star$, $y_{1,\star}(x_0^\star) = 1$, $y_{2,\star}(x_0^\star) = 0$, $y_{3,\star}(x_0^\star) = 0$.
`Output:` A non trivial element in $\ker \tilde{\mathcal{E}}_{D_r}^{N,k}(x_0^\star, y_0^\star)$, if it exists, with minimal weighted degree, or "None".

This subroutine can be reduced to a linear algebra problem. Indeed, we just have to find a polynomial $\mathcal{S}(x, y, y_1, y_2, y_3)$ such that

$$\mathcal{S}\big(x, y_\star(x), \dots, y_{3,\star}(x)\big) = 0 \mod (x - x_0^\star)^\sigma,$$

where $\mathcal{S} \in V_r$ and $\deg_{x,y}(\mathcal{S}) \leq N$.
We recall that in our study V_r is the vector space associated to the extactic curve $\tilde{E}_{D_r}^{N,k}(x_0, y_0)$, see Definition 27, Definition 32 and Definition 37.
We will give some details in Section 7, in order to explain how we can compute $\mathcal{S}$ efficiently.

6.1. **Rational first integrals.** An algorithm which computes a rational first integral with degree smaller than N has been described in [5]. Here, we use the same kind of approach. However, we only use one random point $(x_0^\star, y_0^\star) \in \mathbb{K}^2$. In [5], two random points were used.

Compute Rational first integral
`Input:` $A, B \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$
`Output:` An equation $(Eq_0) : \mathcal{F} - F = 0$ where $F(x,y) \in \mathbb{K}(x,y) \setminus \mathbb{K}$, or "None" or "I don't know".

(1) If $A(x_0^\star, y_0^\star) = 0$ then Return "I don't know".
(2) Compute flow series$(A, B, x_0^\star, y_0^\star, N, 0)$=: $y_\star(x)$.
(3) Compute solution extactic kernel$(A, B, y_\star(x), N, 0)$=:$\mathcal{S}$.
If $\mathcal{S}$="None", then Return "None", else $\mathcal{S} =: P$.
(4) Build Rational first integral$(A, B, P, x_0^\star, y_0^\star)$.

Now, we describe the algorithm Build Rational first integral.

Build Rational first integral
`Input:` $P, A, B \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star) \in \mathbb{K}^2$.

`Output:` An equation $(Eq_0) : \mathcal{F} - F = 0$, where $F(x,y) \in \mathbb{K}(x,y) \setminus \mathbb{K}$, or "I don't know".

(1) Compute $P_{red} = P/\gcd(P, \partial_x P, \partial_y P)$.
(2) Compute the factorization $\gcd\big(P_{red}, D_0(P_{red})\big) = \prod_{j=1}^{l} L_j(x,y)$, where L_j are irreducible in $\mathbb{K}[x,y]$ and set $i := 1$.
(3) While $L_i(x_0^\star, y_0^\star) \neq 0$ do $i := i+1$.
(4) If $i > l$ then Return("I don't know"),
Else
   (a) $\Omega := \dfrac{D_0(L_i)}{L_i}$;
   (b) Compute a basis $\mathcal{B} = \{b_1, b_2, \dots\}$ of the kernel of the linear system $D_0(Q) = \Omega Q$, where $Q \in \mathbb{K}[x,y]_{\leq \deg(L_i)}$.
   (c) If $|\mathcal{B}| = 1$ then Return("I don't know"),
   Else Return$\Big((Eq_0) : \mathcal{F} - \dfrac{b_1}{b_2} = 0\Big)$.

Proposition 48. *The algorithm* Compute Rational first integral *satisfies the following properties:*

- *If it returns "None" then the derivation D_0 has no rational first integral with degree smaller than N.*
- *If it returns an equation (Eq_0) then this equation leads to a rational first integral of D_0.*
- *If it returns "I don't know", then $(x_0^\star, y_0^\star)$ belongs to*
$$\Sigma_0 \cup \mathfrak{S}_N,$$
where $\Sigma_0 = \mathcal{V}(A) \cup \Sigma_{D_0,N}$.

Proof. If the algorithm returns "None", this means that $\tilde{E}_{D_0}^N(x_0^\star, y_0^\star) \neq 0$, thus $\tilde{E}_{D_0}^N(x_0, y_0) \neq 0$. Therefore, Theorem 26 implies that D_0 has no rational first integral.

If the algorithm returns (Eq_0) then by construction we have $D_0(b_i) = \Omega b_i$, for $i = 1, 2$. Thus
$$D_0\Big(\frac{b_1}{b_2}\Big) = \frac{D_0(b_1)b_2 - b_1 D_0(b_2)}{b_2^2} = 0.$$
Furthermore, $\frac{b_1}{b_2} \notin \mathbb{K}$ because b_1 and b_2 are linearly independent. Then, we deduce that $\frac{b_1}{b_2}$ is a rational first integral.

Now, we suppose that $(x_0^\star, y_0^\star) \notin \Sigma_0 \cup \mathfrak{S}_N$ and we are going to prove that the algorithm does not return "I don't know".

First, if $\tilde{E}_{D_0}^N(x_0^\star, y_0^\star) \neq 0$, then as seen before, the algorithm returns the correct output "None".

Second, suppose that $\dim_{\mathbb{K}} \tilde{\mathcal{E}}_{D_0}^N(x_0^\star, y_0^\star) \neq 0$.
In Step 3 of Compute Rational first integral we have: $\mathcal{S} \in \ker \tilde{\mathcal{E}}_{D_0}^N(x_0^\star, y_0^\star)$. As $(x_0^\star, y_0^\star) \notin \Sigma_{D_0,N}$, Lemma 43 implies $\mathcal{S}\big(x, y_\star(x)\big) = 0$. Then an irreducible factor L_i of $\mathcal{S}$ is an irreducible Darboux polynomial which vanishes at $(x_0^\star, y_0^\star)$.
As $(x_0^\star, y_0^\star) \notin \mathfrak{S}_N$, we deduce that there exists a rational first integral R/Q with

degree $p \leq N$ and that the factor L_i of $\mathcal{S}$ is a Darboux polynomial of degree p of the form $\lambda R - \mu Q$, where $Q, R \in \mathbb{K}[x, y]$. Moreover, a rational first integral is defined up to an homography. Then the rational first integral can be written $L_i/Q \in \mathbb{K}(x, y) \setminus \mathbb{K}$, where Q is linearly independent of L_i over $\mathbb{K}$.

We have then

$$D_0\Big(\frac{L_i}{Q}\Big) = 0 \Rightarrow \frac{D_0(L_i)Q - L_i D_0(Q)}{Q^2} = 0 \Rightarrow \frac{D_0(L_i)}{L_i} = \frac{D_0(Q)}{Q}.$$

The last equality defines the polynomial Ω and we remark that the linear system considered in Step 4b of Build Rational first integral has two independent solutions: L_i and Q. Thus the basis $\mathcal{B}$ contains two distinct elements b_1 and b_2. As seen before, these two elements give a rational first integral.
In conclusion, if $(x_0^\star, y_0^\star) \notin \Sigma_0 \cup \mathfrak{S}_N$, the algorithm does not return "I don't know". □

6.2. **Darbouxian first integrals.** This section describes how to use the results given in the previous section in order to get an efficient probabilistic algorithm computing Darbouxian first integrals. The strategy is the following: First, we compute a non trivial element $y_1^k P + Q \in \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_0^\star)$. Second, from P and Q we build a Darbouxian first integral.

Compute Darbouxian first integral
`Input:` $A, B \in \mathbb{K}[x, y]$, $(x_0^\star, y_0^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$, $k \in \mathbb{N}^*$ (by default $k = 1$)
`Output:` An equation $(Eq_1) : \partial_y \mathcal{F} - F^{1/k} = 0$, where $F(x, y) \in \mathbb{K}(x, y) \setminus \{0\}$, or an equation $(Eq_0) : \mathcal{F} - F = 0$ where $F(x, y) \in \mathbb{K}(x, y) \setminus \mathbb{K}$,
or "None" or "I don't know".

(1) If $A(x_0^\star, y_0^\star) = 0$ then Return "I don't know".
(2) Compute flow series$(A, B, x_0^\star, y_0^\star, N, 1)$=: $y_\star(x), y_{1,\star}(x)$.
(3) Compute solution extactic kernel$(A, B, y_\star(x), y_{1,\star}(x), N, 1, k)$=:$\mathcal{S}$.
If $\mathcal{S}$="None", then Return "None", Else $\mathcal{S} =: y_1^k P + Q$.
(4) Build Darbouxian first integral$(A, B, P, Q, x_0^\star, y_0^\star, k)$.

Now, we describe the algorithm Build Darbouxian first integral.

Build Darbouxian first integral
`Input:` $A(x, y), B(x, y)$, $P(x, y), Q(x, y) \in \mathbb{K}[x, y]$ with $(P, Q) \neq (0, 0)$, $(x_0^\star, y_0^\star)$ in $\mathbb{K}^2$, $k \in \mathbb{N}^*$ (by default $k = 1$).
`Output:` An equation $(Eq_1) : \partial_y \mathcal{F} - F^{1/k} = 0$, where $F(x, y) \in \mathbb{K}(x, y) \setminus \{0\}$, or an equation $(Eq_0) : \mathcal{F} - F = 0$ where $F(x, y) \in \mathbb{K}(x, y) \setminus \mathbb{K}$
or "I don't know".

(1) If $P = 0$ then Return(Build Rational first integral$(Q, A, B, x_0^\star, y_0^\star)$).
(2) If $Q = 0$ then Return(Build Rational first integral$(P, A, B, x_0^\star, y_0^\star)$).
(3) $R_1 := APQ(P/Q)^{-1/k}\Big(D_0((P/Q)^{1/k}) + A(P/Q)^{1/k}\partial_y(B/A)\Big)$.
(4) If $R_1 = 0$ then Return $(Eq_1) : \partial_y \mathcal{F} - (P/Q)^{1/k} = 0$,
Else Return(Build Rational first integral $(R_1, A, B, x_0^\star, y_0^\star)$).

Proposition 49. *The algorithm* Compute Darbouxian first integral *satisfies the following properties:*

- *If it returns "None" then the derivation D_0 has no k-Darbouxian nor rational first integral with degree smaller than N.*

- *If it returns an equation* (Eq_0) *or* (Eq_1) *then this equation leads to a first integral of* D_0.
- *If it returns "I don't know", then* $(x_0^\star, y_0^\star)$ *belongs to*

$$\Sigma_1 \cup \mathfrak{S}_{2N+2d-1},$$

where $\Sigma_1 = \mathcal{V}(A) \cup \Sigma_{D_0,N} \cup \Sigma_{D_1,N,k}$.

Proof. If the algorithm returns "None", this means that $\tilde{E}_{D_1}^{N,k}(x_0^\star, y_0^\star) \neq 0$. Theorem 30 implies that D_0 has no rational nor k-Darbouxian first integral with degree smaller than N.

If the algorithm returns (Eq_1) this means that we have $R_1 = 0$ in Build Darbouxian first integral. Proposition 10 gives then the desired result.

If the algorithm returns (Eq_0) then this result comes from Build Rational first integral. We have seen in Proposition 48 that this output is correct.

Now, we prove the last point of the proposition. We suppose that $(x_0^\star, y_0^\star)$ does not belong to $\Sigma_1 \cup \mathfrak{S}_{2N+2d-1}$, and we are going to prove that the algorithm does not return "I don't know".

First, if $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_0^\star) = 0$ then in Step 3 of Compute Darbouxian first integral we have $\mathcal{S} =$"None". Thus the algorithm returns "None".

Second, we suppose that $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_0^\star) \neq 0$.
In Step 3 of Compute Darbouxian first integral we have $\mathcal{S} = y_1^k P + Q$.

• If $P = 0$, then $Q \neq 0$ and $Q \in \ker \tilde{\mathcal{E}}_{D_0}^{N}(x_0^\star, y_0^\star)$ thanks to Lemma 29. Then in Step 1 of Build Darbouxian first integral the algorithm returns an equation (Eq_0) thanks to Proposition 48 and the inclusion $\Sigma_0 \cup \mathfrak{S}_N \subset \Sigma_1 \cup \mathfrak{S}_{2N+2d-1}$.

• If $P \neq 0$ then $Q \neq 0$. Indeed, if $Q = 0$ then as $(x_0^\star, y_0^\star) \notin \Sigma_{D_1,N,k}$ we have, thanks to Lemma 43, $y_{1,\star}(x)^k P\big(x, y_\star(x)\big) = 0$. Since $y_{1,\star}(x) \neq 0$ we deduce that $P\big(x, y_\star(x)\big) = 0$. Therefore a factor $\mathcal{P}$ of P is a Darboux polynomial which vanishes at $(x_0^\star, y_0^\star)$. It would give a non trivial element in $\ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y_0^\star)$. This is absurd since Compute solution extactic kernel returns a solution with minimal weighted degree. It follows $Q \neq 0$.
Furthermore, we have $Q \notin \ker \tilde{\mathcal{E}}_{D_0}^{N}(x_0^\star, y_0^\star)$. Indeed, since $(x_0^\star, y_0^\star) \notin \Sigma_{D_0,N}$, the contrary would imply $Q\big(x, y_\star(x)\big) = 0$, and gives a non trivial element in $\ker \tilde{\mathcal{E}}_{D_1}^{N,k}(x_0^\star, y^\star)$. This is absurd since $y_1^k P + Q$ is a solution with minimal weighted degree.
In Build Darbouxian first integral, we thus compute R_1.
If $R_1 = 0$ then by Proposition 10 we get a Darbouxian first integral.
Now, we suppose that $R_1 \neq 0$, then in Step 4 of Build Darbouxian first integral, we use the algorithm Build Rational first integral.
As $(x_0^\star, y_0^\star) \notin \Sigma_{D_1,N,k}$ we have $y_{1,\star}(x)^k P\big(x, y_\star(x)\big) + Q\big(x, y_\star(x)\big) = 0$ thanks to Lemma 43. Therefore, with the strategy used in Lemma 29, we obtain $R_1\big(x, y_\star(x)\big) = 0$ and then $R_1 \in \ker \tilde{\mathcal{E}}_{D_0}^{2N+2d-1}(x_0^\star, y_0^\star)$. Thus, we deduce that the algorithm returns an equation (Eq_0) giving a rational first integral thanks to Proposition 48 and the

inclusion $\Sigma_0 \cup \mathfrak{S}_{2N+2d-1} \subset \Sigma_1 \cup \mathfrak{S}_{2N+2d-1}$.

In conclusion if $(x_0^\star, y_0^\star) \notin \Sigma_1 \cup \mathfrak{S}_{2N+2d-1}$ the algorithm does not return "I don't know". □

Proposition 50. *We set*

$$\mathcal{D}(d,N) = d + \mathcal{B}_0(d,N) + \mathcal{B}_1(d,N) + \Big(\frac{d(d+1)}{2} + 5\Big)(2N+2d-1).$$

There exists a polynomial H with degree smaller than $\mathcal{D}(d,N)$ such that:
If $H(x_0^\star, y_0^\star) \neq 0$ then Compute Darbouxian first integral *returns "None" or an equation leading to a first integral.*

Proof. From Corollary 45 we deduce the existence of a polynomial $\tilde{H}_1$ such that:

$$\Sigma_{D_0,N} \cup \Sigma_{D_1,N,k} \cup \mathcal{V}(A) \subset \mathcal{V}(\tilde{H}_1)$$

where $\deg(\tilde{H}_1) \leq d + \mathcal{B}_0(d,N) + \mathcal{B}_1(d,N)$. We also have from Lemma 47 the existence of a polynomial $\tilde{H}_2$ of degree $\Big(\frac{d(d+1)}{2} + 5\Big)(2N+2d-1)$ such that:

$$\mathfrak{S}_{2N+2d-1} \subset \mathcal{V}(\tilde{H}_2).$$

Thus the polynomial

$$H(x_0, y_0) = \tilde{H}_1(x_0, y_0)\tilde{H}_2(x_0, y_0)$$

vanishes on the set given in Proposition 49 in the "I don't know" part. So if $H(x_0^\star, y_0^\star) \neq 0$ then Compute Darbouxian first integral returns "None" or an equation leading to a first integral, and the degree of H satisfies the degree bound. □

Corollary 51. *Let Ω a finite subset of $\mathbb{K}$ of cardinal $|\Omega|$ greater than $\mathcal{D}(d,N)$ and assume that in* Compute Darbouxian first integral *$x_0^\star$, $y_0^\star$ are chosen independently and uniformly at random in Ω. Then,* Compute Darbouxian first integral *returns "None" or an equation leading to a first integral with probability at least*

$$1 - \frac{\mathcal{D}(d,N)}{|\Omega|}.$$

Proof. This follows from Proposition 50, and Zippel-Schwartz's lemma, see [21]. □

Remark 52. *In practice, see Section 8, we never obtain the output "I don't know".*

Proposition 53. *If D_0 admits a rational or Darbouxian first integral with degree smaller than N then* Compute Darbouxian first integral *returns an equation with minimal degree.*

Proof. This follows directly from the fact that Compute solution extactic kernel returns a solution with minimal weighted degree. □

6.3. **Liouvillian first integrals.** Here, in order to compute a Liouvillian first integral, we follow the same strategy as before:
First, we compute a non trivial element $P(x,y)y_1^2 + Q(x,y)y_2 + R(x,y)y_1$ in $\ker \tilde{\mathcal{E}}_{D_2}^N(x_0^\star, y_0^\star)$.
Second, we build from P, Q, R a Liouvillian first integral.

Compute Liouvillian first integral
`Input:` $A, B \in \mathbb{K}[x,y]$, $(x_0^\star, y_0^\star) \in \mathbb{K}^2$, $N \in \mathbb{N}$
`Output:` An equation $(Eq_2) : \partial_y^2 \mathcal{F} - F\partial_y \mathcal{F} = 0$, where $F(x,y) \in \mathbb{K}(x,y)$,

or $(Eq_1) : \partial_y \mathcal{F} - F = 0$, where $F(x, y) \in \mathbb{K}(x, y) \setminus \{0\}$
or $(Eq_0) : \mathcal{F} - F = 0$ where $F(x, y) \in \mathbb{K}(x, y) \setminus \mathbb{K}$,
or "None" or "I don't know".

(1) If $A(x_0^\star, y_0^\star) = 0$ then Return "I don't know".
(2) Compute flow series$(A, B, x_0^\star, y^\star, N, 2)$=: $y_\star(x), y_{1,\star}(x), y_{2,\star}(x)$.
(3) Compute solution extactic kernel$(A, B, y_\star(x), y_{1,\star}(x), y_{2,\star}(x), N, 2)$=:$\mathcal{S}$.
If $\mathcal{S}$="None", then Return("None"),
Else $\mathcal{S} =: P(x, y)y_1^2 + Q(x, y)y_2 + R(x, y)y_1$.
(4) Build Liouvillian first integral$(A, B, P, Q, R, x_0^\star, y_0^\star)$.

Build Liouvillian first integral

`Input:` $A(x, y), B(x, y), P(x, y), Q(x, y), R(x, y) \in \mathbb{K}[x, y]$ such that $(P, Q, R) \neq 0$, $(x_0^\star, y_0^\star) \in \mathbb{K}^2$.

`Output:`An equation $(Eq_2) : \partial_y^2 \mathcal{F} - F(x, y)\partial_y \mathcal{F} = 0$, or $(Eq_1) : \partial_y \mathcal{F} - F = 0$, where $F(x, y) \in \mathbb{K}(x, y) \setminus \{0\}$, or $(Eq_0) : \mathcal{F} - F = 0$, where $F(x, y) \in \mathbb{K}(x, y) \setminus \mathbb{K}$, or "I don't know".

(1) If $Q = 0$ then Return(Build Darbouxian first integral$(A, B, P, R, x_0^\star, y_0^\star)$).
(2) Compute $P_1 := A^3Q^2\big(D_0(P/Q) + A(P/Q)\partial_y(B/A) + A\partial_y^2(B/A)\big)$,
$Q_1 := A^3Q^2D_0(R/Q)$.
(3) If $P_1 = 0$ then Return $(Eq_2) : \partial_y^2\mathcal{F} - (P/Q)\partial_y\mathcal{F} = 0$,
Else Return$\big($Build Darbouxian first integral$(A, B, P_1, Q_1, x_0^\star, y_0^\star)\big)$.

Proposition 54. *The algorithm* Compute Liouvillian first integral *satisfies the following properties:*

- *If it returns "None" then the derivation D_0 has no Liouvillian nor Darbouxian nor rational first integral with degree smaller than N.*
- *If it returns an equation (Eq_0) or (Eq_1) or (Eq_2) then this equation leads to a non-trivial first integral.*
- *If it returns "I don't know", then $(x_0^\star, y_0^\star)$ belongs to*
$$\Sigma_2 \cup \mathfrak{S}_{4N+8d-3},$$
where $\Sigma_2 = \mathcal{V}(A) \cup \Sigma_{D_0,N} \cup \Sigma_{D_1,N} \cup \Sigma_{D_2,N}$.

Proof. If the algorithm returns "None", this means that $\tilde{E}_{D_2}^N(x_0^\star, y_0^\star) \neq 0$. Theorem 35 implies that D_0 has no rational nor Darbouxian nor Liouvillian first integral with degree smaller than N.

If the algorithm returns (Eq_2) this means that we have $P_1 = 0$ in Build Liouvillian first integral. Proposition 10 gives then the desired result.

If the algorithm returns (Eq_1) this result is correct thanks to Proposition 49. Indeed, the algorithm returns (Eq_1) when Build Liouvillian first integral uses Build Darbouxian first integral.

If the algorithm returns (Eq_0) then the output is correct as shown in Proposition 48.

Now, we prove the last point of the proposition and we suppose that $(x_0^\star, y_0^\star)$ does not belong to $\Sigma_2 \cup \mathfrak{S}_{4N+8d-3}$.

First, if $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_2}^N(x_0^\star, y_0^\star) = 0$ then in Step 3 of Compute Liouvillian first integral we have $\mathcal{S} =$"None". Thus the algorithm returns "None".

Second, we suppose that $\dim_{\mathbb{K}} \ker \tilde{\mathcal{E}}_{D_2}^N(x_0^\star, y_0^\star) \neq 0$.
In Step 3 of Compute Liouvillian first integral we have $\mathcal{S} = Py_1^2 + Qy_2 + Ry_1$.

• If $Q = 0$ then $Py_1 + R \in \ker \tilde{\mathcal{E}}_{D_1}^N(x_0^\star, y_0^\star)$ as shown in Lemma 34. Then we are in the Darbouxian case. Therefore Proposition 49 and the inclusion $\Sigma_1 \cup \mathfrak{S}_{2N+2d-1} \subset \Sigma_2 \cup \mathfrak{S}_{4N+8d-3}$ allow us to conclude in this situation.

• If $Q \neq 0$ then in Build Liouvillian first integral, we compute P_1.
If $P_1 = 0$ then by Proposition 10 we get a Liouvillian first integral.
Now, we suppose $P_1 \neq 0$.
We claim

$$Q \notin \ker \tilde{\mathcal{E}}_{D_0}^N(x_0^\star, y_0^\star).$$

Indeed, if $Q \in \ker \tilde{\mathcal{E}}_{D_0}^N(x_0^\star, y_0^\star)$ then $Q\big(x, y_\star(x)\big) = 0$ because $(x_0^\star, y_0^\star)$ does not belong to $\Sigma_{D_0,N}$. Thus a factor $\mathcal{R}$ of Q is a Darboux polynomial.
Therefore, $\mathcal{R}$ would give a non-trivial element in $\ker \tilde{\mathcal{E}}_{D_2}^N(x_0^\star, y_0^\star)$. This is impossible since the computed solution $\mathcal{S}$ has a minimal weighted degree and w-deg$(\mathcal{S}) \geq$ w-deg$(\mathcal{R})$.

As $(x_0^\star, y_0^\star) \notin \Sigma_{D_2,N}$, we have thanks to Lemma 43

$$P\big(x, y_\star(x)\big)y_{1,\star}^2(x) + Q\big(x, y_\star(x)\big)y_{2,\star}(x) + R\big(x, y_\star(x)\big)y_{1,\star}(x) = 0.$$

Since $Q \notin \ker \tilde{\mathcal{E}}_{D_0}^N(x_0^\star, y_0^\star)$ we can use the strategy of Lemma 34 and we get

$$P_1\big(x, y_\star(x)\big)y_{1,\star}(x) + Q_1\big(x, y_\star(x)\big) = 0,$$

where $\deg(P_1), \deg(Q_1) \leq 2N + 3d - 1$.
Then, as we have $\Sigma_1 \subset \Sigma_2$, Proposition 49 implies that the algorithm Build Darbouxian first integral applied to P_1 and Q_1 gives either a Darbouxian first integral with degree smaller than $2N + 3d - 1$, or a rational first integral.

Thus when $(x_0^\star, y_0^\star) \notin \Sigma_2 \cup \mathfrak{S}_{4N+8d-3}$ the algorithm does not return "I don't know". □

Proposition 55. *We set*

$$\mathcal{L}(d, N) = d + \mathcal{B}_0(d, N) + \mathcal{B}_1(d, N) + \mathcal{B}_2(d, N) + \Big(\frac{d(d+1)}{2} + 5\Big)(4N + 8d - 3).$$

There exists a polynomial H_L with degree smaller than $\mathcal{L}(d, N)$ such that:
If $H_L(x_0^\star, y_0^\star) \neq 0$ then Compute Liouvillian first integral returns "None" or an equation leading to a first integral.

Proof. The proof is done exactly in the same way as the proof of Proposition 55. □

Corollary 56. *Let Ω a finite subset of $\mathbb{K}$ of cardinal $|\Omega|$ greater than $\mathcal{L}(d, N)$ and assume that in Compute Liouvillian first integral $x_0^\star$, $y_0^\star$ are chosen independently and* *uniformly at random in Ω. Then, Compute Liouvillian first integral returns "None " or an equation leading to a first integral with probability at least*

$$1-\frac{\mathcal{L}(d,N)}{|\Omega|}.$$

Proposition 57. *If D_0 admits a rational or Darbouxian or Liouvillian first integral with degree smaller than N then Compute Liouvillian first integral returns an equation with minimal degree.*

Proof. As in the Darbouxian case this is a direct consequence of the minimality of the weighted degree of a solution in Compute solution extactic kernel. □

6.4. **Riccati first integrals.** In order to compute a Riccati first integral, we follow the same strategy as before:
First, we compute a non trivial element $4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)+R(x,y)y_1^2$ in $\tilde{\mathcal{E}}^N_{D_3}(x_0^\star,y_0^\star)$. Second, we build from P, Q, R a Riccati first integral.

Compute Riccati first integral
`Input:` $A,B\in\mathbb{K}[x,y]$, $(x_0^\star,y_0^\star)\in\mathbb{K}^2$, $N\in\mathbb{N}$
`Output:` An equation $(Eq_3):\partial_y^2\mathcal{F}-F\mathcal{F}$, where $F(x,y)\in\mathbb{K}(x,y)$,
or $(Eq_1):\partial_y\mathcal{F}-\sqrt{F}=0$, where $F(x,y)\in\mathbb{K}(x,y)\setminus\{0\}$,
or $(Eq_0):\mathcal{F}-F=0$, where $F(x,y)\in\mathbb{K}(x,y)\setminus\mathbb{K}$,
or "None" or "I don't know".

(1) If $A(x_0^\star,y_0^\star)=0$ then Return "I don't know".
(2) Compute flow series$(A,B,x_0^\star,y_0^\star,N,3)$=: $y_\star(x),y_{1,\star}(x),y_{2,\star}(x),y_{3,\star}(x)$.
(3) Compute solution extactic kernel$(A,B,y_\star(x),y_{1,\star}(x),y_{2,\star}(x),y_{3,\star}(x),N,3)$=:$\mathcal{S}$.
If $\mathcal{S}$="None", then Return("None"),
else $\mathcal{S}=:4P(x,y)y_1^4+Q(x,y)(3y_2^2-2y_3y_1)+R(x,y)y_1^2$.
(4) Build Riccati first integral$(A,B,P,Q,R,x_0^\star,y_0^\star)$.

Build Riccati first integral
`Input:` $A(x,y),B(x,y)$, $P(x,y),Q(x,y),R(x,y)\in\mathbb{K}[x,y]$ such that $(P,Q,R)\neq 0$, $(x_0^\star,y_0^\star)\in\mathbb{K}^2$.
`Output:`An equation $(Eq_3):\partial_y^2\mathcal{F}-F(x,y)\mathcal{F}=0$, where $F(x,y)\in\mathbb{K}(x,y)$,
or $(Eq_1):\partial_y\mathcal{F}-\sqrt{F(x,y)}=0$, where $F(x,y)\in\mathbb{K}(x,y)\setminus\{0\}$,
or $(Eq_0):\mathcal{F}-F=0$, where $F(x,y)\in\mathbb{K}(x,y)\setminus\mathbb{K}$, or "I don't know".

(1) If $Q=0$ then Return(Build Darbouxian first integral$(A,B,4P,R,x_0^\star,y_0^\star,2)$).
(2) Compute $P_1:=A^4Q^2\big(4D_0(P/Q)+8A(P/Q)\partial_y(B/A)-2A\partial_y^3(B/A)\big)$,
$Q_1:=A^4Q^2D_0(R/Q)$.
(3) If $P_1=0$ then Return $(Eq_3):\partial_y^2\mathcal{F}-(P/Q)\mathcal{F}=0$
Else Return$\big($Build Darbouxian first integral$(A,B,P_1,Q_1,x_0^\star,y_0^\star,2)\big)$.

Proposition 58. *The algorithm Compute Riccati first integral satisfies the following properties:*

- *If it returns "None" then the derivation D_0 has no Riccati nor 2-Darbouxian nor rational first integral with degree smaller than N.*
- *If it returns an equation (Eq_0) or (Eq_1) or (Eq_3) then this equation leads to a non-trivial first integral.*

- *If it returns "I don't know", then* $(x_0^\star, y_0^\star)$ *belongs to*
$$\Sigma_3 \cup \mathfrak{S}_{4N+10d-3},$$
where $\Sigma_3 = \mathcal{V}(A) \cup \Sigma_{D_0,N} \cup \Sigma_{D_1,N,2} \cup \Sigma_{D_3,N}$.

Proof. The proof of this proposition is similar to the one given for Proposition 54. □

As before we deduce the following results:

Proposition 59. *We set*
$$\mathcal{R}(d,N) = d + \mathcal{B}_0(d,N) + \mathcal{B}_1(d,N) + \mathcal{B}_3(d,N) + \Big(\frac{d(d+1)}{2} + 5\Big)(4N + 10d - 3)$$
There exists a polynomial H_R *with degree smaller than* $\mathcal{R}(d,N)$ *such that:*
If $H_R(x_0^\star, y_0^\star) \neq 0$ *then* Compute Riccati first integral *returns "None" or an equation leading to a first integral.*

Proof. The proof is done exactly in the same way as the proof of Proposition 55. □

Corollary 60. *Let* Ω *a finite subset of* $\mathbb{K}$ *of cardinal* $|\Omega|$ *greater than* $\mathcal{R}(d,N)$ *and assume that in* Compute Riccati first integral $x_0^\star$, $y_0^\star$ *are chosen independently and uniformly at random in* Ω. *Then,* Compute Riccati first integral *returns "None" or an equation leading to a first integral with probability at least*
$$1 - \frac{\mathcal{R}(d,N)}{|\Omega|}.$$

Proposition 61. *If* D_0 *admits a rational or 2-Darbouxian or Riccati first integral with degree smaller than* N *then* Compute Riccati first integral *returns an equation with minimal degree.*

Proof. As in the Darbouxian case this is a direct consequence of the minimality of the weighted degree of a solution in Compute solution extactic kernel. □

6.5. **Deterministic algorithms.** In this section we show how to get a deterministic algorithm from our probabilistic ones. We give explicitly the deterministic algorithm for the Riccati case below. The Darbouxian and Liouvillian can be obtained in the same way.

Deterministic computation Riccati first integral
`Input:` $A, B \in \mathbb{K}[x,y]$, such that $A(x,y) \neq 0$, $N \in \mathbb{N}$
`Output:` An equation $(Eq_3) : \partial_y^2 \mathcal{F} - F\mathcal{F}$, where $F(x,y) \in \mathbb{K}(x,y)$,
or $(Eq_1) : \partial_y \mathcal{F} - \sqrt{F} = 0$, where $F(x,y) \in \mathbb{K}(x,y) \setminus \{0\}$,
or $(Eq_0) : \mathcal{F} - F = 0$, where $F(x,y) \in \mathbb{K}(x,y) \setminus \mathbb{K}$,
or "None" or "I don't know".

(1) Set $c := 0$, $x_0^\star := -1$.
(2) While $c \leq \mathcal{R}(d,N) + 1$ do
    (a) $x_0^\star := x_0^\star + 1$, $\Omega := \emptyset$.
    (b) While $A(x_0^\star, y) = 0$ do $x_0^\star := x_0^\star + 1$.
    (c) While $|\Omega| \leq \mathcal{R}(d,N) + 1$ do
        (i) Choose a random element $y_0^\star \in \mathbb{K} \setminus \Omega$ such that $A(x_0^\star, y_0^\star) \neq 0$.
        (ii) $\mathcal{E} :=$Compute Riccati first integral$(A, B, (x_0^\star, y_0^\star), N)$.
        (iii) If $\mathcal{E} =$"None", then Return "None".

(iv) If $\mathcal{E} =$"I don't know" then $\Omega := \Omega \cup \{y_0^\star\}$, Else Return $\mathcal{E}$.
(d) $c := c + 1$.
(3) Return "None".

Proposition 62. *The algorithm* Deterministic computation Riccati first integral *is correct.*

Proof. The deterministic algorithm repeats the probabilistic algorithm. If the probabilistic algorithm returns an equation or "None" then this output is correct thanks to Proposition 58.
Now, we want to get $x_0^\star, y_0^\star$ such that $H_R(x_0^\star, y_0^\star) \neq 0$.
As we use the probabilistic algorithm with at most $\mathcal{R}(d,N)+1$ different values for $x_0^\star$ and $\mathcal{R}(d,N)+1$ different values for $y_0^\star$ we necessarily avoid situations where $H_R(x_0^\star, y_0^\star)$ is equal to zero. Then Proposition 59 implies that the probabilistic algorithm returns an output different from "I don't know" and we get the desired output. □

7. Complexity results

In this section we study the arithmetic complexity of our algorithms. We focus on the dependency on the degree bound N and we recall that we assume that $N \geq d$, where $d = \max(\deg(A), \deg(B))$ denotes the degree of the polynomial vector field. This hypothesis is natural because if a derivation has a polynomial first integral of degree N, then necessarily $N - 1 \geq d$. More precisely, we suppose that d is fixed and N tends to infinity.

All the complexity estimates are given in terms of arithmetic operations in $\mathbb{K}$. We use the notation $f \in \tilde{\mathcal{O}}(g)$, roughly speaking this means that we neglect the logarithmic factors in the expression of the complexity. For a precise definition, see [21, Definition 25.8].
We suppose that the Fast Fourier Transform can be used so that two univariate polynomials with coefficients in $\mathbb{K}$ and degree bounded by r can be multiplied in $\tilde{\mathcal{O}}(r)$, see [21, Corollary 8.19].
We further assume that two matrices of size n with entries in $\mathbb{K}$ can be multiplied using $\mathcal{O}(n^\omega)$, where $2 \leq \omega \leq 3$ is the matrix multiplication exponent, see [21, Chapter 12]. We also recall that a basis of solutions of a linear system composed of m equations and $n \leq m$ over $\mathbb{K}$ can be computed using $\mathcal{O}(mn^{\omega-1})$ operations in $\mathbb{K}$, see [4, Chapter 2].

The algorithm Compute flow series is a direct application of the algorithm given in [6]. In our situation, the number of arithmetic operations needed to perform this subroutine is in $\tilde{\mathcal{O}}(L\sigma + \sigma)$. Here L is the number of arithmetic operations needed to evaluate the rational functions defining the system (S_r'). As d is assumed to be fixed, we have $L \in \mathcal{O}(1)$. Furthermore, σ is the precision on the power series, then $\sigma \in \mathcal{O}(N^2)$.
It thus follows that the computation modulo $(x - x_0^\star)^\sigma$ of $y_\star(x)$, $y_{1,\star}(x)$, $y_{2,\star}(x)$, $y_{3,\star}(x)$ can done with at most $\mathcal{O}(N^2)$ arithmetic operations.

In Compute solution extactic kernel we need to find a non trivial element in $\ker \tilde{\mathcal{E}}_{r,D_r}^N(x_0^\star, y_0^\star)$. This can be done with an Hermite-Padé approximation. We

recall this setting:
We have m polynomials $f_i(x) \in \mathbb{K}[x]$, a precision σ, a shift $s = (s_1, \dots, s_m)$ and we want to compute m polynomials $p_i(x) \in \mathbb{K}[x]$ such that

$$\sum_{i=1}^{m} p_i.f_i = 0 \mod x^{\sigma}.$$

The set of all solutions $(p_1, \dots, p_m)$ is a $\mathbb{K}[x]$-module. A s-minimal approximate basis is a basis of this module and furthermore an element of this basis has minimal s-degree among all solutions of the problem. We recall that the s-degree of $(p_1, \dots, p_m)$ is $\max_i \deg(p_i + s_i)$.
We can compute such a basis with $\tilde{\mathcal{O}}\big(m^{\omega-1}(\sigma+\xi)\big)$ arithmetic operations in $\mathbb{K}$, where $\xi = \sum_i (s_i - \min(s))$, see [3], [38, Theorem 5.3] and [23].
In our situation we have $r \in [[0;3]]$, $m = (r+1)(N+1)$, $\sigma = (r+1)\frac{(N+1)(N+2)}{2}$.

When $r = 0$ we set:

$$(f_1, \dots, f_m) = \big(1, y_\star(x), y_\star^2(x), \dots, y_\star^N(x)\big),$$

$$s = (0, 1, 2, \dots, N).$$

When $r = 1$ we set:

$$\begin{aligned}(f_1, \dots, f_m) \;=\; & \big(1, y_\star(x), y_\star^2(x), \dots, y_\star^N(x),\\ & y_{1,\star}(x), y_{1,\star}(x)y_\star(x), y_{1,\star}(x)y_\star^2(x), \dots, y_{1,\star}(x)y_\star^N(x)\big),\end{aligned}$$

$$s = (0, 1, 2, \dots, N, N+1, \dots, 2N+1).$$

When $r = 2$ we set

$$\begin{aligned}(f_1, \dots, f_m) \;=\; & \big(y_{1,\star}(x), y_{1,\star}(x)y_\star(x), y_{1,\star}(x)y_\star^2(x), \dots, y_{1,\star}(x)y_\star^N(x),\\ & y_{1,\star}^2(x), y_{1,\star}^2(x)y_\star(x), y_{1,\star}^2(x)y_\star^2(x), \dots, y_{1,\star}^2(x)y_\star^N(x),\\ & y_{2,\star}(x), y_{2,\star}(x)y_\star(x), y_{2,\star}(x)y_\star^2(x), \dots, y_{2,\star}(x)y_\star^N(x)\big),\end{aligned}$$

$$s = (0, 1, 2, \dots, N, N+1, \dots, 2N+1, 2N+2, \dots, 3N+2).$$

When $r = 3$ we set

$$\begin{aligned}(f_1, \dots, f_m) \;=\; & \big(y_{1,\star}^4(x), y_{1,\star}^4(x)y_\star(x), y_{1,\star}^4(x)y_\star^2(x), \dots, y_{1,\star}^4(x)y_\star^N(x),\\ & \Psi(x), \Psi(x)y_\star(x), \Psi(x)y_\star^2(x), \dots, \Psi(x)y_\star^N(x),\\ & y_{1,\star}^2(x), y_{1,\star}^2(x)y_\star(x), y_{1,\star}^2(x)y_\star^2(x), \dots, y_{1,\star}^2(x)y_\star^N(x)\big)\end{aligned}$$

where $\Psi(x) = 3y_{2,\star}^2(x) - 2y_{3,\star}(x)y_{1,\star}(x)$, and

$$s = (0, 1, 2, \dots, N, N+1, \dots, 2N+1, 2N+2, \dots, 3N+2).$$

We remark that from a solution $(p_1, \dots, p_m)$ we get:

- when $r = 1$, a polynomial

$$Q(x,y) + P(x,y)y_1 = \sum_{i=0}^{N} p_i(x)y^i + \sum_{i=0}^{N} p_{N+1+i}(x)y^i y_1,$$

- when $r=2$, a polynomial

$$R(x,y)y_1 + P(x,y)y_1^2 + Q(x,y)y_2 = \sum_{i=0}^{N} p_i(x)y^i y_1 + \sum_{i=0}^{N} p_{N+1+i}(x)y^i y_1^2 + \sum_{i=0}^{N} p_{2N+2+i}(x)y^i y_2,$$

- when $r=3$, a polynomial

$$P(x,y)y_1^4 + Q(x,y)\Psi + R(x,y)y_1^2 = \sum_{i=0}^{N} p_i(x)y^i y_1^4 + \sum_{i=0}^{N} p_{N+1+i}(x)y^i \Psi + \sum_{i=0}^{N} p_{2N+2+i}(x)y^i y_1^2,$$

where $\Psi = 3y_2^2 - 2y_3y_1$.

Therefore a solution with a minimal s-degree corresponds to a polynomial in $\ker \tilde{\mathcal{E}}^N_{r,D_r}(x_0^\star, y_0^\star)$ with minimal weighted degree. Thus the subroutine Compute solution extactic kernel can be done with at most $\tilde{\mathcal{O}}(N^{\omega-1}N^2) = \tilde{\mathcal{O}}(N^{\omega+1})$ arithmetic operations in $\mathbb{K}$.

The algorithm Build Rational first integral computes a gcd of bivariate polynomials with degree in $\mathcal{O}(N)$. This subroutine can be done with at most $\tilde{\mathcal{O}}(N^2)$ arithmetic operations in $\mathbb{K}$, see [21]. Furthermore, we need to factorize a bivariate polynomial with degree at most N, this can be done in a probabilistic (respectively deterministic) way with $\tilde{\mathcal{O}}(N^3)$ (respectively $\tilde{\mathcal{O}}(N^{\omega+1})$) arithmetic operations plus the factorization of an univariate polynomial in $\mathbb{K}[T]$ with degree N, see [7, 26].

At last, in Build Rational first integral we solve the linear system

$$D(Q) = \Omega Q, \text{ where } \deg(Q) \leq N.$$

This step can done with $\mathcal{O}(N^{\omega+1})$ arithmetic operations. The approach is the following. We set

$$A(x,y) = \sum_{j=0}^{d} a_j(y/x)x^j, \; B(x,y) = \sum_{j=0}^{d} b_j(y/x)x^j,$$

$$\Omega(x,y) = \sum_{j=0}^{d-1} \omega_{j+1}(y/x)x^j, \; \omega_0 = 0, \; Q(x,y) = \sum_{j=0}^{N} q_{N-j}(y/x)x^j,$$

and we also set $q_j = 0$, for all $j \notin \{0,\dots,N\}$.
We want to solve the equation

$$A\partial_x Q + B\partial_y Q - \Omega Q = 0. \tag{7.1}$$

Substituting in Equation 7.1 the above expressions with $y = zx$, we obtain for the coefficient of $x^{N+d-1-j}$

$$\sum_{i=0}^{d} \big(b_{d-i}(z) - z a_{d-i}(z)\big) q'_{j-i}(z) + \big((N-j+i)a_{d-i}(z) - \omega_{d-i}(z)\big) q_{j-i}(z)$$

Then Equation 7.1 gives

$$\begin{aligned} (7.2) \qquad & \big(za_d(z)-b_d(z)\big)q_j'(z)+\big(\omega_d(z)-(N-j)a_d(z)\big)q_j(z) \\ = \quad & \sum_{i=1}^{d} \big(b_{d-i}(z)-za_{d-i}(z)\big)q_{j-i}'(z)+\big((N-j+i)a_{d-i}(z)-\omega_{d-i}(z)\big)q_{j-i}(z) \end{aligned}$$

So the equation $A\partial_x Q + B\partial_y Q - \Omega Q = 0$ is equivalent to the system of equations (7.2) for $j = 0, \dots, N+d$. This system is triangular as the righthandside of (7.2) only involves q_l with $l < j$. Let us now remark that there can be at most one $j = j_0$ such that

$$za_d(z) = b_d(z), \;\; \omega_d(z) = (N-j)a_d(z)$$

as we cannot have $a_d = b_d = \omega_d = 0$. Thus equation (7.2) seen as a differential equation in q_j always admits an affine space of polynomial solutions of dimension ≤ 1 for $j \neq j_0$.

We now solve the system of equations (7.2) by induction on j. For $j = 0$, equation (7.2) is a linear differential equation of order 1, and thus admits a vector space of polynomial solutions of dimension ≤ 1 if $j_0 \neq 0$ or dimension $N+1$ for $j = j_0$. This vector space can be found by solving a linear system of $N+1$ unknowns and $N+d$ equations, which costs $\mathcal{O}(N^\omega)$.

Let us now assume $(q_{N-j}, \dots, q_N)$ belongs to a known vector space $\mathcal{V}_j$. We look at Equation (7.2) for $j+1$. This is a linear differential equation in $q_{N-j-1} \in \mathbb{K}[z]_{\leq N}$ and $(q_{N-j}, \dots, q_N) \in \mathcal{V}_j$. As it is of order 1 in q_{N-j-1}, the dimension of the space of solutions $\mathcal{V}_{j+1}$ can grow at most by 1 if $j \neq j_0$ or $N+1$ if $j = j_0$. Such linear system with $N+1+\dim \mathcal{V}_j$ unknowns and $N+d$ equations can be solved in $\mathcal{O}\big((N+d)(N+1+\dim \mathcal{V}_j)^{\omega-1}\big)$ operations.

Now, as $\dim \mathcal{V}_j$ grows at most by one except at most for one $j = j_0$ where it grows at most by $N+1$, we always have $\dim \mathcal{V}_j \leq 2N+d$. Thus each step of the resolution of (7.2) costs at most $\mathcal{O}\big((N+d)(N+1+2N+d)^{\omega-1}\big) = \mathcal{O}(N^\omega)$. As there are $N+d$ steps, the overall cost is $\mathcal{O}(N^{\omega+1})$.

In conclusion our probabilistic algorithms use at most $\tilde{\mathcal{O}}(N^{\omega+1})$ arithmetic operations in $\mathbb{K}$ plus the factorization of a univariate polynomial with degree at most N. This is the complexity given in Theorem 4.

As $\mathcal{D}(d,N), \mathcal{L}(d,N)$ and $\mathcal{R}(d,N)$ are in $\mathcal{O}(N^4)$, the deterministic algorithm uses at most $\tilde{\mathcal{O}}(N^{\omega+9})$ arithmetic operations in $\mathbb{K}$ and $\mathcal{O}(N^8)$ factorizations of univariate polynomials in $\mathbb{K}[T]$ with degree at most N.

8. Examples

The algorithms developed in the previous sections have been implemented in Maple. This implementation is available with some examples at:
`http://combot.perso.math.cnrs.fr/software.html`,
`https://www.math.univ-toulouse.fr/~cheze/Programme.html`.
The computations for the following examples have been done on a Macbook pro 2013, intel core i7 2.8 Ghz.
For practical reasons, the implemented version of our algorithms do not use the Hermite-Padé algorithm to find a solution of the extactic kernel. We just solve

a linear system. Furthermore, the solutions $y_\star(x),\dots,\ y_{3,\star}(x)$ are computed from $y_\star(x)$ and then integrated. For example, we compute $y_{1,\star}(x)$ with the formula:

$$y_{1,\star}(x) = exp\Big(\int \frac{B}{A}(x, y_\star(x))dx\Big).$$

8.1. **The Darbouxian case.** Let us consider the system

$$\dot{x} = x^2 + 2xy + y^2 - 4x + 4y - 2,\ \ \dot{y} = x^2 + 2xy + y^2 + 4x - 4y - 2.$$

The algorithm Compute Darbouxian first integral returns in $0.2s$, when $N = 3$:

$$\frac{\partial \mathcal{F}}{\partial y} + \frac{14(x^2 + 2xy + y^2 - 4x + 4y - 2)}{11(x - y)(x^2 + 2xy + y^2 - 2)} = 0,$$

which after integration leads to the Darbouxian first integral

$$\mathcal{F}(x, y) = \sqrt{2}\ln\left(x + y - \sqrt{2}\right) - \sqrt{2}\ln\left(x + y + \sqrt{2}\right) + \ln(x - y).$$

Now, we set

$$z = \frac{x + y + \sqrt{2}}{x + y - \sqrt{2}}, \quad w = x - y,$$

and we have for the level set $\mathcal{F}(x, y) = c$

$$w = e^c z^{\sqrt{2}}.$$

This curve is not algebraic for almost all c, and thus the system does not admit a rational first integral.

The initial point used in the execution of the algorithm above was $(1, 8)$. To get "I don't know", we need for example to use a bad point, i.e. a point vanishing a Darboux polynomial. From this point, we will obtain a Darboux polynomial, and thus the algorithm will try to deduce from this polynomial a rational first integral. This will not work as the vector field has no rational first integral. We can choose for example $(1, 1)$ or $(1, \sqrt{2} - 1)$. Such initial points were never encountered when using (small) random initial points. In particular, the probabilistic algorithm is the only algorithm necessary to use in practice, and we never have to rerun it with several initial points.

If we use the algorithm Compute Darbouxian first integral with $N = 2$ then the output is "None". This is correct and means that there exists no Darbouxian first integral with degree smaller than 2.

Now, we slightly modify the previous example

$$\dot{x} = 2\lambda^2 x - 2\lambda^2 y + \lambda^2 - x^2 - 2xy - y^2,\ \ \dot{y} = 2\lambda^2 y - 2\lambda^2 x + \lambda^2 - x^2 - 2xy - y^2.$$

The algorithm Compute Darbouxian first integral returns with $\lambda = 100$ and $N = 3$

$$\frac{\partial \mathcal{F}}{\partial y} - \frac{312(x^2 + 2xy + y^2 - 20000x + 20000y - 10000)}{469(x - y)(y + 100 + x)(y - 100 + x)} = 0$$

in $0.2s$ which after integration leads to the Darbouxian first integral

$$\mathcal{F}(x, y) = 100\ln(x + y - 100) - 100\ln(x + y + 100) + \ln(x - y).$$

Now the exponential of $\mathcal{F}$ gives a rational first integral

$$\left(\frac{x+y-100}{x+y+100}\right)^{100}(x-y),$$

which is of degree 101.

We remark that if we want to compute a rational first integral we can use Compute Darbouxian first integral. In this case the bound N is a bound on the degree of the product of the irreducible Darboux polynomials used to write the rational first integral (3 in the previous example) and not a bound on the degree of the first integral (101 in the previous example). The difference between these bounds is important when the rational first integral has one or several factors with large multiplicities.

8.2. **Comparison with the Avelar-Duarte-da Mota's algorithm.** We compare our algorithm with the algorithm proposed by J. Avellar, L.G.S. Duarte, L.A.C.P. da Mota, denoted in the following by: ADM algorithm, see [1].
First, we consider a vector field of the form $(-\partial_y G, \partial_x G)$ with

$$G = x + \sum_{i=1}^{m} i \ln(x+y-i),$$

after multiplying by a common denominator.
We find the Darbouxian first integral in the following times.

m	1	2	3	4	5	6
Compute Darbouxian first integral	0.016	0.031	0.063	0.234	0.951	5.1
ADM algorithm	0.094	0.047	0.078	0.109	0.109	0.187

Unexpectedly, the ADM algorithm fails with $m \geq 7$. We see that the ADM algorithm computation times are much better than ours. This is because all Darboux polynomials are of degree 1, and the ADM algorithm computes them first. After there is a combinatorial step, with an exponential complexity, but here it is negligible at those low m.

Secondly, we study a growing degree Darboux polynomial case

$$G = x + \ln(x + y^m - 1).$$

We find the Darbouxian first integral in the following times.

m	1	2	3	4	5	6
Compute Darbouxian first integral	0.015	0.015	0.31	0.125	0.359	0.889
ADM algorithm	0.094	0.078	1.123	$> 10^3$	$> 10^3$	$> 10^3$

The timings of our algorithm have the same order of magnitude, but the ADM algorithm becomes almost unusable. This is because the computation of Darboux polynomials is very expansive even for low degrees. In other words, as soon as the Darboux polynomials are not linear, the ADM algorithm is not usable. Our algorithm never computes Darboux polynomials, and thus avoids this problem.

8.3. **The Liouvillian case.** Consider the system

$$\dot{x} = 2x^2 - 2y^2 - 1,\ \dot{y} = 2x^2 - 2y^2 - 3.$$

The algorithm Compute Liouvillian first integral returns in $0.3s$ when $N = 3$

$$\frac{\partial^2 \mathcal{F}}{\partial y^2} - \frac{2(x+y)(2x^2 - 4xy + 2y^2 - 1)}{2x^2 - 2y^2 - 1}\frac{\partial \mathcal{F}}{\partial y} = 0.$$

After integration, this gives the first integral

$$\mathcal{F}(x,y) = \sqrt{\pi}\mathrm{erf}(x-y) + (x+y)e^{-(x-y)^2}.$$

Now, we set $z = x - y, w = x + y$, and the equation $\mathcal{F}(x,y) = c$ gives

$$w = (c - \sqrt{\pi}\mathrm{erf}(z))e^{z^2}$$

This function is never algebraic for $c \in \mathbb{C}$ as erf is not even elementary. Thus the system does not admit a rational first integral.
The function $\mathcal{F}$ is holomorphic and all solution curves (outside the straight line at infinity) are of the form $\mathcal{F}(x,y) = c \in \mathbb{C}$. As none of these curves are algebraic, the system does not admit any Darboux polynomial. As the poles of a Darbouxian first integral are Darboux polynomials, a Darbouxian first integral should be a polynomial, which is again not possible as there are no rational first integrals. Thus the system admits a Liouvillian first integral but no first integral of lower class.

Now, we consider the example 185 of Kamke which is an Abel equation with a Liouvillian first integral

$$\dot{x} = -x^7,\ \dot{y} = y^2(5x^3 + 2x^2y + 2y).$$

Compute Liouvillian first integral gives in 1.95s with $N = 7$:

$$\frac{\partial^2 \mathcal{F}}{\partial y^2} + \frac{x^6 + 7x^3y + 6x^2y^2 + 6y^2}{2y(x^6 + 2x^3y + x^2y^2 + y^2)}\frac{\partial \mathcal{F}}{\partial y} = 0.$$

This system admits a lower class first integral, a 4-Darbouxian first integral of degree 32, which can be recovered by integration of this equation, giving

$$\tilde{\mathcal{F}}(x,y) = \int \frac{y^{3/2}\sqrt{x}(5x^3 + 2x^2y + 2y)}{(x^6 + 2x^3y + x^2y^2 + y^2)^{5/4}}dx + \frac{x^{15/2}}{(x^6 + 2x^3y + x^2y^2 + y^2)^{5/4}\sqrt{y}}dy.$$

This integral can also be searched directly as a 4-Darbouxian first integral with $N = 32$, and is obtained in $22166s$.

8.4. **The Riccati case.** The example 43 of Kamke is an Abel equation. When we set $a = 3$ and $b = 17$ in this equation, we get

$$\dot{x} = 1,\ \dot{y} = -(9x^2 + 36x + 17)y^3 - 3xy^2.$$

This equation admits a Riccati first integral.
Compute Riccati first integral gives in $10.9s$ with $N = 9$:

$$\frac{\partial^2 \mathcal{F}}{\partial y^2} - \frac{3P}{4(9x^2y + 36xy + 17y - 6)^2y^3}\mathcal{F} = 0,$$

with

$$\begin{aligned} P &= 81x^4y^3 + 648x^3y^3 - 18x^3y^2 + 1602x^2y^3 - 180x^2y^2 + 1224xy^3 \\ &\quad +3x^2y - 466xy^2 + 289y^3 + 24xy - 204y^2 + 36y - 2. \end{aligned}$$

This equation together with the equation of the first integral defines a PDE system with a two dimensional space of solutions. The first integral in Kamke's book is written using Bessel functions. Thus the solutions of this PDE system can be expressed in terms of Bessel functions here, but this is not an easy task.

In general, Abel equations are of the form

$$\frac{\partial y}{\partial x} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x).$$

These can be seen as a generalization of the Riccati equation. However, in contrary to the Riccati equation, they are not all solvable in algebraic-differential terms. Still many integrable families are known, Abel integrability is typically searched by looking into a known table list up to some transformations. Our algorithm can detect any integrable cases, even belonging to an unknown new integrable family.

8.5. **The generic case.** In a generic situation a vector field has no symbolic first integral. Let us now consider a random quadratic vector field

$$\dot{x} = 2x^2 + xy - 2y^2 - 1, \;\; \dot{y} = 2x^2 - 2y^2 + y - 3.$$

We do not find any Liouvillian nor Riccati first integrals (and thus neither Darbouxian, 2-Darbouxian or rational first integral) up to degree 9, with the following timings.

N	1	2	3	4	5	6	7	8	9
Liouvillian	0.015	0.047	0.620	0.219	0.639	1.576	4.072	8.736	17.44
Riccati	0.015	0.047	0.094	0.280	0.843	2.262	4.898	11.62	23.32

8.6. **Rational first integral with degree bigger than N.** Let us consider the following example

$$\dot{x} = \lambda x^3 - \lambda xy^2 - 2\mu y^2 - \lambda x, \;\; \dot{y} = \lambda x^2 y - \lambda y^3 - 2\mu xy - \lambda y$$

with $\lambda, \mu \in \mathbb{Z}$. This vector field always admits the first integral

$$I(x,y) = \lambda \ln\left(\frac{x}{y} - \frac{\sqrt{x^2-y^2}}{y}\right) + \mu \ln\left(\frac{x^2-y^2+1}{x^2-y^2-1} - 2\frac{\sqrt{x^2-y^2}}{x^2-y^2-1}\right)$$

which is a 2-Darbouxian first integral, which is of degree 8. Indeed, we have: $\partial_y I - F = 0$, where $F^2 = P/Q$ and

$$\begin{aligned} P &= \lambda^2 x^6 - 2\,\lambda^2 x^4 y^2 + \lambda^2 x^2 y^4 - 4\,\lambda\,\mu\, x^3 y^2 + 4\,\lambda\,\mu\, xy^4 - 2\,\lambda^2 x^4 + 2\,\lambda^2 x^2 y^2 \\ &\quad +4\,\mu^2 y^4 + 4\,\lambda\,\mu\, xy^2 + \lambda^2 x^2 \\ Q &= x^6 y^2 - 3\,x^4 y^4 + 3\,x^2 y^6 - y^8 - 2\,x^4 y^2 + 4\,x^2 y^4 - 2\,y^6 + x^2 y^2 - y^4. \end{aligned}$$

As $\lambda/\mu \in \mathbb{Q}$, we can however build from this a rational first integral (with degree depending on λ/μ). This is also a particular case of a Liouvillian first integral, which is then of degree 8. This kind of example is build by searching radical extension of $\bar{\mathbb{K}}(x,y)$ with groups of unit of rank ≥ 2. The first integral is then a linear combination of logs of these units.

For this example, we here display the timings in seconds of the algorithms Rational, Darbouxian, Liouvillian and Ricatti first integrals with initial point $(2,5)$.

The degree columns are the minimum N for which the output is not “None”.

(λ,μ)	Rat deg	time	D deg	time	L deg	time	Ric deg	time
(1,0)	1	0.031	1	0.016	1	0.016	1	0.016
(0,1)	2	0.016	1	0.015	1	0.016	2	0.031
(1,1)	3	0.031	2	0.031	2	0.078	3	0.328
(2,1)	4	0.031	3	0.047	2	0.062	3	0.281
(1,2)	5	0.031	4	0.110	4	0.343	5	1.934
(3,1)	5	0.047	4	0.234	3	0.203	5	2.168
(1,3)	7	0.203	5	0.483	5	1.935	7	16.27
(4,1)	6	0.109	5	0.437	4	0.733	5	2.652
(3,2)	7	0.219	5	0.359	5	1.825	6	6.012
(2,3)	8	0.359	6	1.030	5	1.981	6	12.43
(1,4)	9	1.482	6	2.559	6	11.62	8	74.1*
(5,1)	7	0.453	5	0.873	5	3.853	7	26.59
(1,5)	11	3.807	7	4.290	7	20.87	8	54.2*
(6,1)	8	1.264	6	2.980	6	15.14	7	40.62
(5,2)	9	0.889	6	1.326	6	6.381	7	20.37
(4,3)	10	3.354	7	6.443	6	15.50	7	39.57
(3,4)	11	4.945	7	5.694	6	12.76	8	73*
(2,5)	12	8.392	8	9.267	7	30.60	8	83*
(1,6)	13	10.99	8	10.59	8	55.08	8	64.1*

In many cases (all except those with $\star$), Darbouxian, Liouvillian and Ricatti algorithms have returned the rational first integral even if it is of degree larger than N. For example, with $(\lambda, \mu) = (3, 4)$ the Liouvillian algorithm with $N = 6$ returns a rational first integral of degree 11. Remark that the degree cannot be higher than 8 for Liouvillian or Ricatti first integrals, because the 2-Darbouxian first integral is always present and its degree is not growing.

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