On cluster points of alternating projections

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Abstract

Suppose that A and B are closed subsets of a Euclidean space such that $A \cap B \neq \varnothing$, and we aim to find a point in this intersection with the help of the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ generated by the *method of alternating projections*. It is well known that if A and B are convex, then $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to some point in $A \cap B$. The situation in the nonconvex case is much more delicate. In 1990, Combettes and Trussell presented a dichotomy result that guarantees either convergence to a point in the intersection or a nondegenerate compact continuum as the set of cluster points.

In this note, we construct two sets in the Euclidean plane illustrating the continuum case. The sets A and B can be chosen as countably infinite unions of closed convex sets. In contrast, we also show that such behaviour is impossible for finite unions.

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1 Motivation

Let X be a real Euclidean space, and let A and B be closed subsets of X. Our aim is to find a point in $A \cap B$ which we assume to be nonempty. One classical algorithm is the *method of alternating projections*: Given a starting point $b_{-1} \in X$, generate sequences

$$(\forall n \in \mathbb{N}) \quad a_n \in P_A(b_{n-1}) \text{ and } b_n \in P_B(a_n) \tag{1}$$

where $P_C x := \{c \in C \mid \|x - c\| = d_C(x) := \inf_{y \in C} \|x - y\|\}$ denotes the *projection* of x onto C. When A and B are convex, then the projectors P_A and P_B are single-valued and the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to some point in $A \cap B$. This classical result goes back to Bregman [4], and it has found a huge number of extensions (see, e.g., [1], [6], [8], [9]). In the general case, when A and B are not necessarily convex, the situation is much more delicate. In their 1990 paper [7], Combettes and Trussell gave quite general sufficient conditions for the following dichotomy: either $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to a point in $A \cap B$ or the set of cluster points is a nondegenerate continuum. (For recent results in the nonconvex case, see [2] and [3] and the references therein.)

The goal of this note is to explicitly construct two sets A and B illustrating the continuum case.

The sets A and B may be chosen to be countably infinite unions of closed convex sets. In contrast, we also prove that the continuum case cannot occur when A and B are finite unions of closed convex sets.

The remainder of the paper is organized as follows. In Section 2, we lay the ground work by studying a certain curve in the Euclidean plane. In Section 3, we use this curve to construct a sequence of points in the plane that is crucial in obtaining the sets A and B. Some remarks and the announced positive result conclude the paper.

2 An intriguing curve

We will mostly work in the Euclidean plane $\mathbb{R}^2$. As usual, angles will be measured in radians, but sometimes we shall use degrees as in writing $\pi/2 = 90^\circ$.

Let us recall that the distance d between $(r\cos(\alpha), r\sin(\alpha))$ and $(s\cos(\beta), s\sin(\beta))$, where $r \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$, satisfies

$$d^2 = \|(r\cos(\alpha), r\sin(\alpha)) - (s\cos(\beta), s\sin(\beta))\|^2 = r^2 + s^2 - 2rs\cos(\alpha - \beta) \tag{2a}$$

$$\geq r^2 + s^2 - 2rs = (r - s)^2; \tag{2b}$$

hence,

$$r - d \le s \le r + d. \tag{3}$$

Define the function ρ by

$$\rho\colon \mathbb{R}_+ \to \mathbb{R}_+ \colon t \mapsto 1 + \exp(-t). \tag{4}$$

This function will represent the distance of a point on the curve at time t to the origin. Clearly, ρ is strictly decreasing with $\rho(0) = 2$ and $\lim_{t\to+\infty} \rho(t) = 1$. Also define

$$\varepsilon\colon \mathbb{R}_+ \to \mathbb{R}_{++} \colon t \mapsto \frac{\rho(t) - \rho(t + 2\pi)}{2}. \tag{5}$$

Then $\varepsilon' = -\varepsilon$ and hence ε is strictly decreasing to $\lim_{t\to+\infty} \varepsilon(t) = 0$. Note that

$$\mathbb{R}_+ \to \mathbb{R}_{++} \colon \alpha \mapsto \frac{\varepsilon(\alpha)}{\rho(\alpha)} = \frac{1}{2}\frac{1 - e^{-2\pi}}{1 + e^{\alpha}} \text{ is strictly decreasing.} \tag{6}$$

We now define the curve

$$x\colon \mathbb{R}_+ \to \mathbb{R}^2 \colon \alpha \mapsto \rho(\alpha) \cdot \big(\cos(\alpha), \sin(\alpha)\big). \tag{7}$$

Note that x describes a spiral traversing counter-clockwise; x is *injective* because ρ is strictly decreasing. Now let α and β be in $\mathbb{R}_+$, and assume that $\|x(\alpha) - x(\beta)\| \le \varepsilon(\alpha)$. By (3), $\rho(\alpha) - \varepsilon(\alpha) \le \rho(\beta) \le \rho(\alpha) + \varepsilon(\alpha)$. Using the definitions, we solve these inequality for β and obtain

$$\alpha - 0.40 \approx \alpha + \ln(2) - \ln(3 - e^{-2\pi}) \le \beta \le \alpha + \ln(2) - \ln(1 + e^{-2\pi}) \approx \alpha + 0.69; \tag{8}$$

in degrees, this implies $\alpha - 24^\circ \le \beta \le \alpha + 40^\circ$. To summarize,

$$\|x(\alpha) - x(\beta)\| \le \varepsilon(\alpha) \quad \Rightarrow \quad \alpha - 24^\circ \le \beta \le \alpha + 40^\circ. \tag{9}$$

We will now discuss the monotonicity of the function

$$f\colon t \mapsto \|x(\alpha + t) - x(\alpha)\|^2. \tag{10}$$

Because of the triangle inequality (or since $\sin(t) + \cos(t) = \sqrt{2}\sin(t + \pi/4)$), it is clear that

$$t \in \left]0, \pi/2\right[\quad \Rightarrow \quad \sin(t) + \cos(t) > 1. \tag{11}$$

One checks that

$$f'(t)\frac{\exp(2(\alpha + t))}{2} = g_1(t) + g_2(t) + g_3(t), \tag{12}$$

where

$$g_1(t) = \sin(t)\exp(2t+\alpha)(1+\exp(\alpha)), \tag{13a}$$

$$g_2(t) = \exp(\alpha+t)\big(\sin(t)+\cos(t)-1\big), \tag{13b}$$

$$g_3(t) = \exp(t)\big(\sin(t)+\cos(t)-\exp(-t)\big). \tag{13c}$$

Since each g_i is strictly positive on $]0,\pi/2[$, it follows from the mean value theorem that

$$f \text{ is strictly increasing on } [0,\pi/2]. \tag{14}$$

Combining with (9), we deduce[1]

$$\big(\forall\alpha\in\mathbb{R}_+\big)\big(\exists\,!\,\beta>\alpha\big)\ \ \|x(\beta)-x(\alpha)\| = \varepsilon(\alpha). \tag{15}$$

Furthermore, denoting the unit sphere by S, we have

$$(\forall\alpha\in\mathbb{R}_+)\quad d_S(x(\alpha)) = \rho(\alpha)-1 = \exp(-\alpha) > \varepsilon(\alpha). \tag{16}$$

3 An intriguing sequence

We now construct a sequence $(x_n)_{n\in\mathbb{N}}$ in the Euclidean plane with remarkable properties. Let us initialize

$$\alpha_0 := 0,\quad x_0 := x(\alpha_0),\quad \rho_0 := \rho(\alpha_0),\quad \varepsilon_0 := \varepsilon(\alpha_0). \tag{17}$$

In Cartesian coordinates, $x_0 = (2,0)$, and $\varepsilon_0 \approx 0.5$. Now suppose $n\in\mathbb{N}$ and α_n, x_n, ρ_n, and ε_n are given. In view of (15), there exists a unique $\beta > \alpha_n$ such that

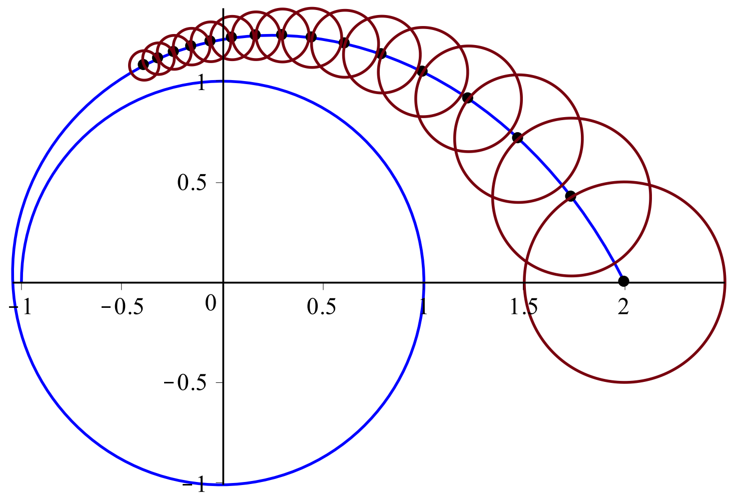
$$\|x(\beta)-x(\alpha_n)\| = \varepsilon_n. \tag{18}$$

We then update

$$\alpha_{n+1} := \beta,\ x_{n+1} := x(\alpha_{n+1}),\ \rho_{n+1} := \rho(\alpha_{n+1}),\ \text{and}\ \varepsilon_{n+1} := \varepsilon(\alpha_{n+1}). \tag{19}$$

(The picture illustrates the beginning of the spiral and $x_0,\ldots,x_{15}$ along with the radii used to construct the next iterate.)

[1] "$\exists\,!$" stands for "there exists a *unique*"



We also set

$$\delta_n := \alpha_{n+1} - \alpha_n. \tag{20}$$

By construction,

$$(\forall n \in \mathbb{N}) \quad \|x_n - x_{n+1}\| = \varepsilon_n \quad \text{and} \quad \sum_{k=0}^{n} \delta_k = \alpha_{n+1} - \alpha_0. \tag{21}$$

Note that

(22) $(\alpha_n)_{n\in\mathbb{N}}$ is strictly increasing, and $(\varepsilon_n)_{n\in\mathbb{N}}$ is strictly decreasing

because the function ε is strictly decreasing. Set

$$\alpha_\infty := \lim_{n\in\mathbb{N}} \alpha_n \in \left]0, +\infty\right]. \tag{23}$$

Since ρ is strictly decreasing we also note that

(24) $(\rho_n)_{n\in\mathbb{N}}$ is strictly decreasing, with $\lim_{n\in\mathbb{N}} \rho_n =: \rho_\infty \in \left[1, 2\right[$.

Hence the corresponding sequence of quotients satisfies

$$1 > q_n := \frac{\rho_{n+1}}{\rho_n} \to 1. \tag{25}$$

Using (2a) and the half-angle identity for sine, we have

$$
\begin{aligned}
&\text{(26a)} && (\forall n \in \mathbb{N}) \quad \varepsilon_n^2 = \|x_n - x_{n+1}\|^2 \\
&\text{(26b)} && = \rho_n^2 + \rho_{n+1}^2 - 2\rho_n\rho_{n+1}\cos(\delta_n) \\
&\text{(26c)} && = (\rho_n - \rho_{n+1})^2 + 2\rho_n\rho_{n+1}\big(1 - \cos(\delta_n)\big) \\
&\text{(26d)} && = (\rho_n - \rho_{n+1})^2 + 4\rho_n\rho_{n+1}\frac{1 - \cos(\delta_n)}{2} \\
&\text{(26e)} && = (\rho_n - \rho_{n+1})^2 + 4\rho_n\rho_{n+1}\sin^2(\delta_n/2).
\end{aligned}
$$

Dividing by ρ_n^2 and recalling (6), we obtain

$$(\forall n \in \mathbb{N}) \quad \left(\frac{1}{2}\frac{1 - e^{-2\pi}}{1 + e^{\alpha_n}}\right)^2 = \frac{\varepsilon_n^2}{\rho_n^2} = (1 - q_n)^2 + 4q_n \sin^2(\delta_n/2). \tag{27}$$

Taking limits, we learn that

$$\left(\frac{1}{2}\frac{1 - e^{-2\pi}}{1 + e^{\alpha_\infty}}\right)^2 = 4\lim_n \sin^2(\delta_n/2). \tag{28}$$

Since δ_n, in degrees, belongs to $]0^\circ, 40^\circ]$ by (9), we deduce that $(\delta_n)_{n\in\mathbb{N}}$ is convergent as well. If $\alpha_\infty = +\infty$, then $\delta_n \to 0$ by (28); however, if $\alpha_\infty < +\infty$, then $\delta_n = \alpha_{n+1} - \alpha_n \to \alpha_\infty - \alpha_\infty = 0$. Hence, we *always* must have

$$\delta_n \to 0. \tag{29}$$

Again by (28), we have

$$\alpha_n \to \alpha_\infty = +\infty, \tag{30}$$

which by (21) implies

$$\sum_{n\in\mathbb{N}} \delta_n = +\infty, \tag{31}$$

$$\varepsilon_n \to 0, \tag{32}$$

and

$$\rho_n \to \rho_\infty = 1. \tag{33}$$

Note also that in view of (26), we have

$$\varepsilon_n^2 > 4\sin^2(\delta_n/2) \geq \frac{\delta_n^2}{4} \quad \text{eventually}, \tag{34}$$

where we used (29) and the Taylor estimate

$$\sin(t/2) \geq \frac{1}{2}t - \frac{1}{48}t^3 = \frac{t}{2}\Big(1 - \frac{1}{24}t^2\Big) \geq \frac{t}{4} \quad \text{for } t \text{ sufficiently close to } 0. \tag{35}$$

Combining with (31), we record that

$$(\forall n \in \mathbb{N}) \quad \|x_n - x_{n+1}\| > \|x_{n+1} - x_{n+2}\| \to 0, \quad \text{and} \quad \sum_{n\in\mathbb{N}} \|x_n - x_{n+1}\| = +\infty. \tag{36}$$

Furthermore, (30) and (33) imply that

$$\text{the set of cluster points of } (x_n)_{n\in\mathbb{N}} \text{ is the unit sphere } S. \tag{37}$$

Define

$$(\forall n \in \mathbb{N}) \quad C_n := \{x_0, x_1, \ldots\} \smallsetminus \{x_n\} \tag{38}$$

We claim that

$$(\forall n \in \mathbb{N}) \quad P_{C_n} x_n = \{x_{n+1}\}. \tag{39}$$

Let $n \in \mathbb{N}$. Since $D_n := \{x_{n+1}, x_{n+2}, \ldots\} \subset x\big(\,]\alpha_n, +\infty[\,\big)$, it follows from (9), (14), and (15) that $P_{D_n} x_n = \{x_{n+1}\}$. We show that there is no $k \in \mathbb{N}$ such that $k < n$ and $\|x_k - x_n\| < \|x_n - x_{n+1}\|$. Suppose the contrary. Then, by (9), $\alpha_n - 24^\circ \leq \alpha_k < \alpha_n$. Hence $\alpha_k < \alpha_n \leq \alpha_k + 24^\circ$. By (14), $\|x_k - x_{k+1}\| = \|x(\alpha_k) - x(\alpha_{k+1})\| \leq \|x(\alpha_k) - x(\alpha_n)\| = \|x_k - x_n\| < \|x_n - x_{n+1}\| < \|x_k - x_{k+1}\|$, which is absurd. This verifies (39). Furthermore, by (16),

$$(\forall n \in \mathbb{N}) \quad d_S(x_n) > \|x_n - x_{n+1}\|. \tag{40}$$

Let us summarize our findings.

Theorem 3.1 *The sequence $(x_n)_{n\in\mathbb{N}}$ and the set $Y := \{x_n \mid n \in \mathbb{N}\}$ satisfy the following:*

(i) $(\|x_n - x_{n+1}\|)_{n\in\mathbb{N}}$ *is strictly decreasing.*

(ii) $x_n - x_{n+1} \to 0$.

(iii) $\sum_{n\in\mathbb{N}} \|x_n - x_{n+1}\| = +\infty$.

(iv) $(\forall n \in \mathbb{N})\ P_{(S\cup Y)\smallsetminus\{x_n\}} x_n = \{x_{n+1}\}$.

(v) *The set of cluster points of $(x_n)_{n\in\mathbb{N}}$ is the compact continuum S.*

(vi) *$S \cup D$ is closed, where D is an arbitrary subset of Y.*

We now obtain the announced example concerning an instance of the method of alternating projections whose set of cluster points is a nondegenerate compact continuum.

Corollary 3.2 *Set* $A := \{x_{2n} \mid n \in \mathbb{N}\} \cup S$*,* $B := \{x_{2n+1} \mid n \in \mathbb{N}\} \cup S$*, and* $b_{-1} := x_0$*. Then* A *and* B *are nonempty compact subsets of* $\mathbb{R}^2$*. The corresponding sequences of alternating projections satisfy*

$$(\forall n \in \mathbb{N}) \quad a_n = P_A b_{n-1} = x_{2n} \text{ and } b_n = P_B a_n = x_{2n+1}. \tag{41}$$

Moreover, $a_n - b_{n-1} \to 0$*,* $b_n - a_n \to 0$*, and* S *is the set of cluster points of* $(a_n)_{n\in\mathbb{N}}$ *and of* $(b_n)_{n\in\mathbb{N}}$*.*

Remark 3.3 Some comments on Corollary 3.2 are in order.

(i) We note that Corollary 3.2 is the first example constructed where the set of limit points of alternating projections is a nondegenerate compact continuum. This complements the analysis of Combettes and Trussell [7] who conceived this case.

(ii) If the starting point b_{-1} is an arbitrary point, then either $a_0 \in S$ or $a_0 \in A \smallsetminus S$. In the first case, we have $(\forall n \in \mathbb{N})\ a_n = b_n = a_0$; in the second case, the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are tails of $(x_{2n})_{n\in\mathbb{N}}$ and $(x_{2n+1})_{n\in\mathbb{N}}$ respectively. A more involved analysis shows that if b_{-1} is outside the closed unit disk, then $P_A b_{-1} \in A \smallsetminus S$ and we are in the second case. Hence one obtains a nondegenerate compact continuum of cluster points exactly when b_{-1} lies outside the closed unit disk.

(iii) The conclusion of Corollary 3.2 hold also true if we replace S be the closed unit disk. In this case, both A and B are *countably infinite* unions of convex sets. In the following result, we show that a degenerate continuum cannot occur as the set of cluster points when A and B are *finite* unions of nonempty closed convex sets.

Theorem 3.4 (finite unions of convex sets) *Suppose that* I *and* J *are nonempty finite index sets, let* $(A_i)_{i\in I}$ *and* $(B_j)_{j\in J}$ *be families of nonempty closed convex subsets of a Euclidean space* X*, and set* $A := \bigcup_{i\in I} A_i$ *and* $B := \bigcup_{j\in J} B_j$*. Consider a sequence of alternating projections* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *generated by* A *and* B*:* $b_{-1} \in X$*, and* $(\forall n \in \mathbb{N})\ a_n \in P_A b_{n-1}$ *and* $b_n \in P_B a_n$*. Suppose that* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are bounded, and that* $b_n - a_n \to 0$ *and* $a_{n+1} - b_n \to 0$*. Then there exists a point* $c \in A \cap B$ *such that* $a_n \to c$ *and* $b_n \to c$*.*

Proof. After relabeling and considering the tails of the sequences if necessary, we assume that each A_i and each B_j is projected upon infinitely often. The pigeonhole principle gives $(i_+, j_+) \in I \times J$ and subsequences $(a_{k_n})_{n\in\mathbb{N}}$ and $(b_{k_n})_{n\in\mathbb{N}}$ lying in A_{i_+} and B_{j_+} respectively. After passing to further subsequences if necessary, we also assume that there is $c \in A_{i_+} \cap B_{j_+}$ such that $a_{k_n} \to c$ and $b_{k_n} \to c$. Set $I_- := \{i \in I \mid c \notin A_i\}$, $I_+ := I \smallsetminus I_-$,

$J_- := \{j \in J \mid c \notin B_j\}$, $J_+ := J \smallsetminus J_-$, $\delta := \min\{\min_{i\in I_-} d_{A_i}(c), \min_{j\in J_-} d_{B_j}(c), 1\}$, $A_- := \bigcup_{i\in I_-} A_i$, and $B_- := \bigcup_{j\in J_-} B_j$. Since $a_{k_n} \to c$, there exists $m \in \mathbb{N}$ such that $\|a_m - c\| < \delta/2$. Then $d_{B_-}(a_m) \geq d_{B_-}(c) - \|a_m - c\| > \delta - \delta/2 = \delta/2 > \|a_m - c\| \geq d_{B\smallsetminus B_-}(a_m)$. Hence $(\forall j \in J_-)\ b_m \notin P_{B_j}(a_m)$ and similarly $(\forall i \in I_-)\ a_{m+1} \notin P_{A_i}(b_m)$. Thus, $b_m \in \{P_{B_j}(a_m) \mid j \in J_+\}$ and $a_{m+1} \in \{P_{A_i}(b_m) \mid i \in I_+\}$. Therefore, because the projectors are nonexpansive, $\delta/2 > \|a_m - c\| \geq \|b_m - c\| \geq \|a_{m+1} - c\| \geq \cdots$ and recalling the assumption that all sets are projected upon yields $I_- = J_- = \varnothing$, i.e., $c \in \bigcap_{i\in I} A_i \cap \bigcap_{j\in J} B_j$. Since c is a cluster point of $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, it thus follows that $\|a_n - c\| \to 0$ and $\|b_n - c\| \to 0$. ■

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References

[1] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, 2011.

[2] H.H. Bauschke, D.R. Luke, H.M. Phan, and X. Wang, Restricted normal cones and the method of alternating projections: theory, *Set-Valued and Variational Analysis*, in press.

[3] H.H. Bauschke, D.R. Luke, H.M. Phan, and X. Wang, Restricted normal cones and the method of alternating projections: applications, *Set-Valued and Variational Analysis*, in press.

[4] L.M. Bregman, The method of successive project for finding a common point of convex sets, *Soviet Mathematics Doklady* 6 (1965), 688–692.

[5] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces*, Springer, 2012.

[6] Y. Censor and S.A. Zenios, *Parallel Optimization*, Oxford University Press, 1997.

[7] P.L. Combettes and H.J. Trussell, Methods of successive projections for finding a common point of sets in metric spaces, *Journal of Optimization Theory and Applications* 67(3) (1990), 487–507.

[8] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, 1990.

[9] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.