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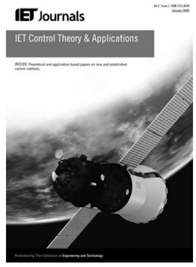

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# An extension of the linear quadratic Gaussian-loop transfer recovery procedure

*L. Ravanbod*[1] *D. Noll*[1] *P. Apkarian*[2]

[1] *Université Paul Sabatier, Institut de Mathématiques, 118 route de Narbonne, 31062 Toulouse, France*
[2] *ONERA, Department of Systems Control and Flight Dynamics, 2 Avenue Edouard Bélin, 31055 Toulouse, France*
*E-mail: LalehRavanbod@yahoo.fr*

**Abstract:** The linear quadratic Gaussian-loop transfer recovery procedure is a classical method to desensibilise a system in closed loop with respect to disturbances and system uncertainty. Here an extension is discussed, which avoids the usual loss of performance in LTR, and which is also applicable for non-minimum phase systems. It is also shown how the idea can be extended to other control structures. In particular, it is shown how proportional integral derivative controllers can be desensibilised with this new approach. The method is tested on several examples, including in particular the lateral flight control of an F-16 aircraft.

## 1 Introduction

It became apparent during the late 1960s that linear quadratic Gaussian (LQG) controllers often lack robustness with regard to system uncertainty. In 1966 Kwakernaak [1] proposed loop transfer recovery (LTR) as a means to overcome this deficit in practical situations. LTR was later re-discovered and popularised in a series of papers by Stein and Athans [2], Doyle and Stein [3, 4]. Even today LQG-LTR is still used by practitioners to desensibilise LQG controllers to enhance the robustness of a design.

Unfortunately, LTR has three main limitations. First, the price for the enhanced robustness may be a considerable loss of performance. Second, LTR is limited to controllers with observer structure. And third, its application to non-minimum phase systems is not obvious. Here we propose a new method, which avoids these difficulties. Our new approach can be cast as a constraint optimisation program offering a trade-off between performance and robustness

$$\begin{array}{ll} \text{minimise} & \mathcal{P}(K) \\ \text{subject to} & \mathcal{R}(K) \leq r \\ & K \text{ structured controller} \end{array} \tag{1}$$

where $\mathcal{P}(K)$ is the performance of the closed-loop system, expressed by an $H_2$ norm, whereas $\mathcal{R}(K)$ is the robustness, represented by a possibly frequency weighted $H_\infty$ norm of the input or output sensitivity function $\|(I+KG)^{-1}\|_\infty$ or $\|(I+GK)^{-1}\|_\infty$. The crucial point is to choose the degree of robustness $r$ in the constraint in such a way that a satisfactory compromise is achieved. As we shall show, for minimum phase systems and observer-based controllers, the LQG-LTR procedure allows us to calibrate $r$ in (1) in a natural way. The mixed $H_2/H_\infty$-controller obtained by solving (1) is then as robust as the corresponding LQG-LTR controller, but has better performance.

In the case of non-minimum phase systems program (1) remains fully in effect. What needs to be modified is the LQG-LTR procedure, at least if one still wishes to use it to calibrate $r$. This can be done, for example, by working with frequency-weighted sensitivity functions. For more details see [5] and special issue on LTR of the International Journal of Robust and Nonlinear Control, especially [6]. In [2] it is also shown that a similar trade-off between sensitivity and complementary sensitivity can be cast as an optimisation problem over the Hardy space of stable transfer functions with 2-norm, that is, an $H_2$- optimisation problem, which under some restrictions can be solved by LQG-LTR.

For more general controller structures, program (1) can be used in much the same way, but one needs a new way to calibrate the robustness parameter $r$ in the constraint. We present a general method that provides a range $[r_*, r^*]$ in which the parameter $r$ should be chosen. The validity of our method is tested for the proportional integral derivative (PID) controller structure.

The structure of the paper is as follows. In Sections 2 and 3, the essential features of LQG-LTR are recalled, presented for the case of the input loop breaking point. The improved LTR procedure for this case is presented in Section 4. Section 5 briefly discusses LTR at the output loop breaking point. Section 6 gives a dual mathematical programming approach, where the roles between performance and robustness in the trade-off are changed. More general controller structures are discussed in Section 7, and a new procedure to calibrate $r$ is introduced. Experiments are presented in Section 8.

## 2 Preparation

Let us briefly recall the set-up for $H_2$-synthesis. Given an open-loop plant in state-space form

$$P : \begin{bmatrix} \dot{x} \\ z_2 \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & B_2 & B \\ \hline C_2 & 0 & D_{2u} \\ C & D_{y2} & 0 \end{array}\right] \begin{bmatrix} x \\ w_2 \\ u \end{bmatrix} \quad (2)$$

the goal of $H_2$ synthesis is to find a dynamic output feedback controller in state-space form

$$K : \begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \begin{bmatrix} x_K \\ y \end{bmatrix} \quad (3)$$

which stabilises $P$ in closed loop and minimises the $H_2$ norm (cf. [7])

$$\min_K \|T_{w_2 \to z_2}(P, K)\|_2 \quad (4)$$

of the closed-loop performance channel $w_2 \to z_2$. We call $\mathcal{P}(K) = \|T_{w_2 \to z_2}(P, K)\|_2$ the performance of the closed-loop system. It is well known that the optimal solution $K^*$ of (4) has observer-based structure

$$K^* = \left[\begin{array}{c|c} A - B_2K_c - K_fC_2 & K_f \\ \hline -K_c & 0 \end{array}\right] \quad (5)$$

and that $K_f$, $K_c$ can be computed via AREs or LMIs [8]. In order to assure the existence of $K^*$ we use standard assumption like (i)–(iv) on page 384 of [8], or (A1)–(A5) on page 387 of [9], which include stabilisability and detectability of the plant (2).

It is convenient to consider LQG control as a special case of $H_2$ synthesis. Following [9], consider the LQG problem

$$G_{LQG} : \begin{cases} \dot{x} = Ax + Bu + \Gamma w \\ y = Cx + v \end{cases}$$

where $w$ and $v$ are white noise with covariance matrices $W$ and $V$, respectively. Let $Q = Q^\top \succeq 0$ and $R = R^\top \succ 0$ and build a plant of form (2) by setting

$$P_{LQG} = \left[\begin{array}{c|c|c} A & B_2 & B \\ \hline C_2 & 0 & D_{2u} \\ \hline C & D_{y2} & 0 \end{array}\right] = \left[\begin{array}{c|cc|c} A & (\Gamma W \Gamma^\top)^{1/2} & 0 & B \\ \hline Q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{1/2} \\ \hline C & 0 & V^{1/2} & 0 \end{array}\right] \quad (6)$$

If the original inputs $v, w$ and outputs $x, u$ of LQG are encoded as $w_2$ and $z_2$ and recovered from the relations

$$\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} W^{1/2} & 0 \\ 0 & V^{1/2} \end{bmatrix} w_2, \quad z_2 = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

then LQG becomes a special case of $H_2$-synthesis in the sense that

$$J = E\left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T (x(t)^\top Qx(t) + u(t)^\top Ru(t))\mathrm{d}t \right\} = \|F_l(P_{LQG}, K^*)\|_2^2$$

for the LQG controller $K^*$. This confirms that the optimal LQG controller $K^*$ has the observer structure (5). The plant $P_{LQG}$ satisfies the standard assumptions for controller synthesis if $(A, (\Gamma W \Gamma^\top)^{1/2}, C)$ and $(A, B, Q^{1/2})$ are assumed stabilisable and detectable [9, 10].

## 3 Loop transfer recovery

This section continues with a rapid flashback on the LQG-LTR procedure [10, 11]. Using the embedding $P_{LQG} \to P$, the situation is interpreted in the context of $H_2$ optimal control.

Along with its excellent performance $p^* = \mathcal{P}(K^*) = \|T_{w_2 \to z_2}(P, K^*)\|_2$, the optimal LQG controller $K^*$ may be highly sensitive and therefore lack robustness with respect to system uncertainty. This is where the LQG-LTR procedure sets in. In its input-sensitivity form it provides a one-parameter family of observer-based controllers

$$K(\rho) = \left[\begin{array}{c|c} A - B_2K_c - K_f(\rho)C_2 & K_f(\rho) \\ \hline -K_c & 0 \end{array}\right]$$

indexed by $0 < \rho \le 1$, such that

1. $K(\rho)$ is the LQG controller of the modified LQG plant

$$P_{LQG}(\rho) = \left[\begin{array}{c|cc|c} A & (\Gamma W \Gamma^\top)^{1/2} & 0 & B \\ \hline Q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{1/2} \\ \hline C & 0 & \rho^{1/2}V^{1/2} & 0 \end{array}\right] \quad (7)$$

the nominal case (6) being $\rho = 1$. In particular, $K^* = K(1)$. Explicitly

$$K(\rho) = -K_c(sI - (A - BK_c - K_f(\rho)C))^{-1}K_f(\rho) \quad (8)$$

2. As $\rho \to 0$, the LTR controller $K(\rho)$ gets less and less sensitive in so far as the $H_\infty$ norm of the LQG sensitivity function $S(G, K(\rho)) = (I + K(\rho)G)^{-1}$ approaches the $H_\infty$ norm of the so-called target sensitivity function $S_{LQ} = (I + K_cG_{LQ})^{-1}$, which has provable good gain and phase margins [12]. Here

$$G(s) = C(sI - A)^{-1}B$$
$$S_{LQ} = (I + K_c(sI - A)^{-1}B)^{-1} = (I + K_cG_{LQ})^{-1}$$

3. $\rho^{1/2}K_f(\rho) \to V^{-1/2}$ as $\rho \to 0$, so $K(\rho)$ has no limit in controller space as $\rho \to 0$. In consequence, performance of $K(\rho)$ degrades in the sense that $\mathcal{P}(K(\rho)) = \|T_{w_2 \to z_2}(P, K(\rho))\|_2 \to \infty$ as $\rho \to 0$, where $P$ is the nominal plant (6).

Altogether the family of LTR controllers $K(\rho)$ in (8) represents a trade-off between performance (4) with respect to the original LQG plant (6), and robustness with respect to the input sensitivity function $S(G, K) = (I + KG)^{-1}$. Each $K(\rho)$ is conveniently obtained by solving a modified LQG synthesis program based on (7). The procedure leaves $K_c$ fixed and adapts the Kalman filter gain $K_f(\rho)$ to the noise level $\rho V$.

*Remark 1:* A variant of the described LTR procedure is obtained by fixing $V = V_0$ and letting $W = W_0 + \rho^{-1}BB^\mathrm{T}$, where $W_0$ is nominal.

The quest addressed in this paper is now how to prove robustness $\|S(G, K)\|_\infty \to \|S_{LQ}\|_\infty =: r_*$ just as in LTR, but at the same time avoid the loss of performance $\mathcal{P}(K(\rho)) \to \infty$ caused by the LTR controller.



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## 4 Improved LQG-LTR procedure

In order to emphasise the terms performance and robustness, we continue to use the notations

$$\mathcal{P}(K) = \|T_{w_2 \to z_2}(P, K)\|_2, \quad \mathcal{R}(K) = \|S(G, K)\|_\infty$$

As was observed before, $\mathcal{R}(K(\rho)) \to r_* := \|S_{LQ}\|_\infty$, while $\mathcal{P}(K(\rho)) \to \infty$ when $\rho \to 0$. Note that $r_*$ is the best robustness we can possibly achieve, so it serves as a lower bound for the parameter $r$ in (1).

Let $r^* := \|S(G, K^*)\|_\infty = \mathcal{R}(K^*)$ be the robustness of the nominal $H_2$ (respectively LQG) controller $K^*$. As $K^*$ is too sensitive with regard to $S(G, K)$, the value $r^*$ is too large. So $r^*$ is an upper bound for $r$. Now every intermediate value $r$ with $r_* < r \le r^* = \mathcal{R}(K^*)$, can be realised as $r = r(\rho) = \mathcal{R}(K(\rho))$ for some $\rho \in (0, 1]$. In other words, for every $r \in (r_*, r^*]$ we can find an LQG-LTR controller $K(\rho)$, which has precisely the robustness $r$.

Naturally, one aims at a compromise $r = r(\rho)$ somewhere in between the two extrema $r_*, r^*$. This is now where LQG-LTR has its limitations. Namely, it can only propose to stop at some $K(\rho)$, where $r = r(\rho)$ is as desired. However, it can then no longer influence the corresponding performance $p(\rho) = \mathcal{P}(K(\rho))$. The value $p(\rho) := \mathcal{P}(K(\rho))$ is just somewhere in between the lower bound $p^* = \mathcal{P}(K^*)$ and the upper bound $p_* = \infty$, and has to be accepted as such. The present work claims that one can do better. Having identified the appropriate robustness level $r = r(\rho) = \mathcal{R}(K(\rho))$ of the LTR controller $K(\rho)$, the following structured mixed $H_2/H_\infty$ optimisation program, a special instance of (1), is proposed.

$$(P_\rho) \quad \begin{array}{ll} \text{minimise} & \mathcal{P}(K) = \|T_{w_2 \to z_2}(P, K)\|_2 \\ \text{subject to} & \mathcal{R}(K) = \|S(G, K)\|_\infty \le r(\rho) \\ & K \text{ has observer structure (5)} \end{array} \quad (9)$$

Its decision variable is $\mathbf{x} = (\text{vec}(K_c), \text{vec}(K_f))$. For the following, the solution of (9) is denoted as $K_{2,\infty}(\rho)$, indicating that a mixed $H_2/H_\infty$ synthesis problem is solved. The robustness level $r(\rho) = \mathcal{R}(K(\rho))$ imposed in the constraint is taken to be the robustness level of the LQG-LTR controller (8) with parameter $\rho$. Program (9) is the key element of Algorithm 1 in Fig. 1.

**Algorithm 1**

1: **Initialise.** Synthesize nominal LQG controller $K^*$ and compute its robustness $r^* = \mathcal{R}(K^*) = \|S(G, K^*)\|_\infty$. If $r^*$ is small enough, meaning that $K^*$ is sufficiently robust, then quit. Otherwise continue.
2: **Calibrate.** Compute LTR controller $K(\rho)$ so that robustness $r(\rho) := \|S(G, K(\rho))\|_\infty < r^*$ is small enough. A lower bound is $r_* = \|S(G_{LQ}, K_c)\|_\infty$.
3: **Optimise.** For the current value $\rho$, solve mixed $H_2/H_\infty$ program $(P_{r(\rho)})$, using $K(\rho)$ as initial guess. The locally optimal solution is $K_{2,\infty}(\rho)$.
4: **Evaluate.** If $K_{2,\infty}(\rho)$ is not sufficiently robust, use smaller $\rho$ to get a smaller $r(\rho)$. If $K_{2,\infty}(\rho)$ is too robust and not sufficiently performing, use larger $\rho$ to get a larger $r(\rho)$. Then go back to Step 3.

**Fig. 1** *Algorithm 1: Trade-off between robustness and performance for LQG*

*Remark 2:* Note that in (9) the Kalman gain $K_f$ and the state feedback gain $K_c$ are optimised simultaneously. The principle of separation of observation and control is no longer valid. In particular, the optimal $K_c$, $K_f$ are no longer characterised by algebraic Riccati equations (AREs). Nonetheless $K_{2,\infty}(\rho)$ is an observer-based controller. Note that without the structural constraint (5) the $H_2/H_\infty$ program (9) has an infinite-dimensional solution [13], which need not even be realisable. And even when realisability is imposed as the sole structural constraint, the optimal solution need not be observer-based.

*Remark 3:* The fact that the $r(\rho)$ cover the range $(r_*, r^*]$ does not mean that $r(\rho) \in (r_*, r^*]$ for all $\rho$. Typically, for $\rho$ close to the nominal value 1 it may happen that $r(\rho) > r^*$. This means LTR is not a monotone procedure, as can be seen from the graph of $100r(\rho)$ in Fig. 2. Naturally, the $\rho$ with $r(\rho) > r^*$ are of no use in Algorithm 1 in Fig. 1. Similarly, for a given $r$ only the largest $\rho$ with $r = r(\rho)$ is of interest.

The central property of the solution $K_{2,\infty}(\rho)$ of (9) is the following:

*Proposition 1:* The optimal $H_2/H_\infty$ controller $K_{2,\infty}(\rho)$ computed in Step 3 of Algorithm 1 in Fig. 1 is as robust as the LTR controller $K(\rho)$ in the sense that $\|S(G, K_{2,\infty}(\rho))\|_\infty = \|S(G, K(\rho))\|_\infty$, but it has better performance $\mathcal{P}(K_{2,\infty}(\rho)) \le \mathcal{P}(K(\rho))$.

*Proof:* The first part of the statement claims that the constraint $\mathcal{R}(K) \le r(\rho)$ in (9) is active at the locally optimal solution $K_{2,\infty}(\rho)$. Suppose this is not the case, that is, $\mathcal{P}(K_{2,\infty}(\rho)) < r(\rho)$. Then $K_{2,\infty}(\rho)$ is also a local minimum of the unconstrained $H_2$ program (4). However, program (4) is strictly convex and its unique global minimum is the LQG controller $K^*$. In particular, there are no other local minima, hence $K^* = K_{2,\infty}(\rho)$. This implies $r^* = \mathcal{P}(K^*) < \mathcal{P}(K(\rho)) = r(\rho)$. However, according to Step 2 of Algorithm 1 in Fig. 1, $\rho$ is such that $r_* < r(\rho) \le r^*$ and values $\rho$ with $r(\rho) > r^*$ are not considered. This shows that the constraint is active.

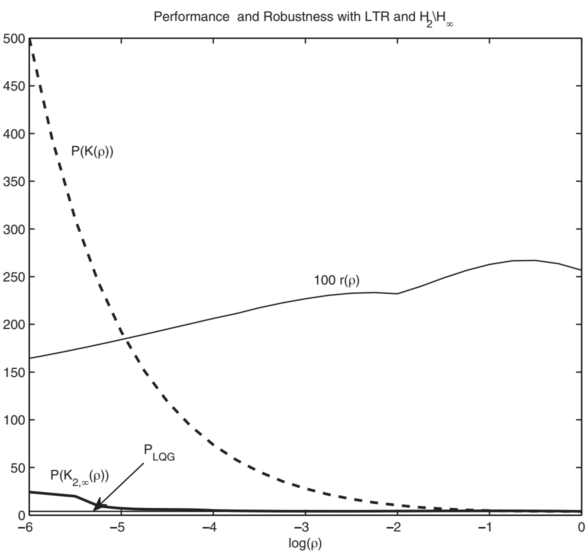


**Fig. 2** *LQG-LTR study: performance of $K(\rho)$ and $K_{2,\infty}(\rho)$ in logarithmic scale*

Lower bound is the performance of the nominal LQG controller. The curve $100r(\rho)$ shows the robustness profile over the same abscissa. As a by-product, it can be seen that LTR is not a monotone procedure

The second claim, the improvement of the performance, is due to the fact that $K(\rho)$ is a feasible point in (9) and that optimisation is started at $K(\rho)$. This assures that the (locally) optimal solution $K_{2,\infty}(\rho)$ has a lower objective value $\mathcal{P}(K_{2,\infty}(\rho)) \leq \mathcal{P}(K(\rho))$. □

*Remark 4:* Mixed $H_2/H_\infty$-programs had originally been proposed by Haddad and Bernstein [14], who characterise the solution in the full-order case (in the absence of constraint (5)) by a system of coupled algebraic Riccati equations. A homotopy method is proposed to compute the solutions. The first numerically efficient way to solve (9) with the constraint (5) was presented in [15] and is based on non-smooth optimisation techniques. Tables 8.3 and 8.4 of [15] give a comparison between the method of Haddad and Bernstein and ours in cases where both are applicable. Note that program (9) is no longer convex owing to the structural constraint on $K$.

## 5 Other LTR procedures

There exists a dual-LTR procedure, which generates a family $K(q)$ of LQG controllers parametrised by $q \geq 0$ such that $K(0) = K^*$, and such that $K(q)$ now gets less sensitive as $q \to \infty$ [3]. Consider the deformed LQG system

$$P(q) : \left[\begin{array}{c|cc|c} A & (\Gamma W \Gamma^\top)^{1/2} & 0 & B \\ \hline Q^{1/2}(q) & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{1/2} \\ \hline C & 0 & V^{1/2} & 0 \end{array}\right]$$

where $Q(q) = Q + qC^\top C$, and $q = 0$ corresponds to the nominal case (6). The LQG-LTR controller is then obtained by an LQG synthesis for $P(q)$ and has the form

$$K(q) = \left[\begin{array}{c|c} A - B_2 K_c(q) - K_f C_2 & K_f \\ \hline -K_c(q) & 0 \end{array}\right] \tag{10}$$

where now $K_f$ is fixed and $K_c(q)$ tuned. Limiting results now hold with respect to the output sensitivity function $\tilde{S}(G, K) = (I + GK)^{-1}$. Namely $\|\tilde{S}(G, K(q))\|_\infty \to \|\tilde{S}_{LQ}\|_\infty$, where $\tilde{S}_{LQ} = (I + C(sI - A)^{-1} K_f)^{-1} = (I + G_{LQ} K_f)^{-1}$, which again has guaranteed margins as $q \to \infty$.

*Remark 5:* Note that $K(q)$ is obtained by artificially increasing the cost term $x^\top Q x$ in the LQG objective, replacing the nominal $Q$ by $Q + qC^\top C$. As $q \to \infty$ increases, this obviously forces the trajectories $x(t)$ to decay faster to 0 as $t \to \infty$, hence a gain in robustness. In [2] a variant is discussed, where in the cost term $x^\top Q x + \mu u^\top R u$ the parameter $\mu$ is driven to zero.

The new type of controller $K_{2,\infty}(q)$ associated with the family $K(q)$ is constructed as follows: Fix $q > 0$ and compute $\tilde{r}(q) = \|\tilde{S}(G, K(q))\|_\infty$. Then solve the mixed $H_2/H_\infty$ program

$$\begin{array}{ll} \text{minimise} & \mathcal{P}(K) = \|T_{w_2 \to z_2}(P, K)\|_2 \\ \text{subject to} & \mathcal{R}(K) = \|\tilde{S}(G, K)\|_\infty \leq \tilde{r}(q) \\ & K \text{ observer-based} \end{array} \tag{11}$$

the solution being $K_{2,\infty}(q)$. The link between the dual LQG-LTR controller $K(q)$ and its associated $H_2/H_\infty$ controller $K_{2,\infty}(q)$ is the following:

*Proposition 2:* The mixed $H_2/H_\infty$ controller $K_{2,\infty}(q)$ is as robust as the LQG-LTR controller $K(q)$ in the sense that $\|\tilde{S}(G, K_{2,\infty}(q))\|_\infty = \|\tilde{S}(G, K(q))\|_\infty$, but it has better performance.

*Remark 6:* It is straightforward to propose an algorithm similar to Algorithm 1 in Fig. 1 based on (11). The details are left to the reader.

## 6 Trade-off with performance certificate

There is a second approach to (9), which can be interpreted as setting aside some of the good performance in order to buy some robustness. Suppose the unconstrained $H_2$ program has $p^* = \mathcal{P}(K^*)$, where $K^*$ solves (4). We refer to $p^*$ as the nominal performance. As soon as $K^*$ is overly sensitive and lacks robustness, $p^*$ is too small. Assuming that we are working with the sensitivity function $\mathcal{R}(K) = \|S(G, K)\|_\infty$, let us consider the following mixed $H_\infty/H_2$ program

$$(D_\alpha) \quad \begin{array}{ll} \text{minimise} & \mathcal{R}(K) = \|S(G, K)\|_\infty \\ \text{subject to} & \mathcal{P}(K) = \|T_{w_2 \to z_2}(P, K)\|_2 \leq (1 + \alpha) p^* \\ & K \text{ has observer structure (5)} \end{array} \tag{12}$$

Here we accept a loss of $100\alpha\%$ in nominal performance $p^*$, and use this freedom to buy as much robustness as possible.

It turns out that there is a close relationship between programs $(P_\rho)$ and $(D_\alpha)$.

*Proposition 3:* Let $K_{2,\infty}(\rho)$ be a Karush–Kuhn–Tucker (KKT) solution of $(P_\rho)$, where $r(\rho)$ is such that the LQG controller is not feasible for $(P_\rho)$. Then there exists $\alpha = \alpha(\rho)$ such that $K_{2,\infty}(\rho) = K_{\infty,2}(\alpha(\rho))$, that is, $K_{2,\infty}(\rho)$ is also a KKT solution of a suitable program $(D_{\alpha(\rho)})$. One simply has to set $\alpha(\rho) := [\mathcal{P}(K_{2,\infty}(\rho)) - p^*]/p^*$.

Conversely, let $K_{\infty,2}(\alpha)$ be a KKT solution of $(D_\alpha)$, which is not a critical point of $\mathcal{R}$ alone and is more robust than the LQG controller. Then $K_{\infty,2}(\alpha) = K_{2,\infty}(\rho(\alpha))$ for a suitable $\rho = \rho(\alpha)$, that is, $K_{\infty,2}(\alpha)$ is also a KKT of $(P_{\rho(\alpha)})$. One has $r(\rho(\alpha)) = \mathcal{R}(K_{\infty,2}(\alpha))$.

*Proof:*
1. Let $K := K_{2,\infty}(\rho)$ be a KKT-point of $(P_\rho)$, respectively, of (9). Then there exists a Lagrange multiplier $\lambda \geq 0$ and a Clarke subgradient $\Phi \in \partial\mathcal{R}(K)$ such that (see [16, Ch. 6])

$$(\text{KKT})_\rho \quad \begin{array}{l} 0 = \nabla\mathcal{P}(K) + \lambda\Phi, \quad \lambda\,(\mathcal{R}(K) - r(\rho)) = 0 \\ \mathcal{R}(K) \leq r(\rho) \end{array}$$

We argue that $\lambda > 0$. Suppose we had $\lambda = 0$. Then $\nabla\mathcal{P}(K) = 0$. By convexity of the LQG program $K$ is then the unique minimum of $\mathcal{P}$, which means it is the LQG controller $K^*$. On the other hand, $\mathcal{R}(K) \leq r(\rho)$ by $(\text{KKT}_\rho)$, which means the LQG controller $K^*$ is feasible in $(P_\rho)$. Since this was excluded by hypothesis, we have a contradiction, proving $\lambda > 0$.

Let us now compare this with the KKT-condition for program $(D_\alpha)$, that is, for (12). Note that $\widetilde{K} := K_{\infty,2}(\alpha)$ is a KKT-point of $(D_\alpha)$ if there exists a subgradient $\Phi \in \partial\mathcal{R}(\widetilde{K})$ and a Lagrange multiplier $\mu \geq 0$ such that

$$(\text{KKT})_\alpha \quad \begin{array}{l} 0 = \Phi + \mu\nabla\mathcal{P}(\widetilde{K}), \quad \mu(\mathcal{P}(\widetilde{K}) - (1 + \alpha)p^*) = 0 \\ \mathcal{P}(\widetilde{K}) \leq (1 + \alpha)p^* \end{array}$$



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All we have to do now is tune $\alpha$ and $\mu$ such that $K$ also satisfies ($\text{KKT}_\alpha$). We simply let $\mu = 1/\lambda$, then the first equation of both conditions is the same. For the constraint, all we have to do is choose $\alpha$ such that $\mathcal{P}(K) = (1+\alpha)p^*$. This is possible, because as we have seen, $K$ is not the LQG controller, hence it satisfies $\mathcal{P}(K) > p^*$. Therefore $\alpha(\rho) = \mathcal{P}(K)/p^* - 1$ as claimed.
2. Conversely, let $\widetilde{K} := K_{\infty,2}(\alpha)$ be a KKT-point of ($D_\alpha$). Then condition ($\text{KKT}_\alpha$) is satisfied. We argue that $\mu > 0$. Indeed, $\mu = 0$ gives $0 = \Phi \in \partial\mathcal{R}(\widetilde{K})$, which means that $\widetilde{K}$ is a critical point of $\mathcal{R}$. Since this was excluded by hypothesis, we must have $\mu > 0$.

Now we have to fix $\rho$ and $\lambda$ in such a way that $\widetilde{K}$ satisfies $(\text{KKT})_\rho$. We simply put $\lambda = 1/\mu$, then the first equations is satisfied. For the constraint, let us put $\tilde{r} := \mathcal{R}(\widetilde{K})$. Then $\tilde{r} < r^* = r(1)$, as by hypothesis $\widetilde{K}$ is more robust than the LQG controller. Since $\tilde{r} > r_*$, and since the curve $r(\rho)$ fills the interval $(r_*, r^*]$, there exists $\rho$ such that $\tilde{r} = r(\rho)$, hence $\mathcal{R}(\widetilde{K}) = r(\rho)$. This $\rho$ is our $\rho(\alpha)$. □

*Remark 7:* While programs ($P_\rho$) and ($D_\alpha$) are at least locally in one-to-one correspondence via $\rho \mapsto \alpha(\rho)$ and $\alpha \mapsto \rho(\alpha)$, it is beneficial to have both at our disposition. For instance, in some cases it may be easier to calibrate the value $\alpha$, that is, the accepted loss of performance, than to guess an appropriate $\rho$ in ($P_\rho$). On the other hand, LTR can be used more directly to calibrate the procedure in the primal approach based on ($P_\rho$). Note, however, a difference between ($D_\alpha$) and ($P_\rho$). In ($D_\alpha$) it may happen that the constraint $\mathcal{P} \leq (1+\alpha)p^*$ is inactive. In that case a local minimum of the robustness function $\mathcal{R}$ alone is found. This is possible, because the $H_\infty$-program $\min\{\|S(G,K)\|_\infty :$ $K$ observer-based$\}$ is not a convex program and may therefore have local minima.

*Remark 8:* The LQG-LTR procedure encounters difficulties for non-minimum phase systems $G$. The target sensitivity function $S_{\text{LQ}}$ can no longer be approached at all frequencies, and a weaker result of the form $S(G, K(\rho)) \to S_{\text{LQ}}(I+E)$ for some frequency-dependent error term $E$ holds instead [5]. In this situation, it may be advantageous to work with weighted sensitivity functions $W_1SW_2$ or $W_1\tilde{S}W_2$ in order to preserve some of the properties of LTR in the minimum phase case, as proposed in [2]. In contrast, program (9), respectively, it dual (12), do not really depend on $G$ being minimum phase. For instance, in (12) we have still interest to minimise sensitivity as much as we can, non-minimum phase being just a warning that we might be less successful. In general we may decide to follow Athans [17] and use LTR despite the limitations of non-minimum phase, or we could use a robustness constraint of the form $\mathcal{R}(K) = \|W_1S(K)W_2\|_\infty \leq r$, respectively, $\mathcal{R}(K) = \|W_1\tilde{S}(K)W_2\|_\infty \leq r$, using a frequency-weighted sensitivity function within a modified LTR procedure to calibrate $r(\rho)$. A third possibility is to use the method proposed in the next section to calibrate the robustness parameter $r$ differently.

## 7 Extension to more general controller structures

In this section, we propose an extension of Algorithm 1 in Fig. 1 to general controller structures. In Section 8.1.2 this will be applied to controllers with PID structure.

A controller in state-space form (3) is called structured if the matrices $A_K$, $B_K$, $C_K$, $D_K$ depend smoothly on a design parameter vector $\mathbf{x}$, that is

$$A_K = A_K(\mathbf{x}), \quad B_K = B_K(\mathbf{x}), \quad C_K = C_K(\mathbf{x}), \quad D_K = D_K(\mathbf{x})$$

It is assumed that $\mathbf{x}$ varies in some parameter space $\mathbb{R}^n$, or in a constrained subset of $\mathbb{R}^n$. Here $n = \dim(\mathbf{x})$ is typically smaller than $\dim(K) = n_K^2 + m_2n_K + p_2n_K + m_2p_2$, where $m_2$ is the number of inputs, $p_2$ the number of outputs, $n_K$ the order of $K$. It is also expected that $n_K \ll n_x$. Full order controllers are en abus de langue referred to as unstructured.

A first controller structure was already encountered, namely, observer-based controllers, where $\mathbf{x} = (\text{vec}(K_c), \text{vec}(K_f)) \in \mathbb{R}^{n_xm_2+n_xp_2}$. Other useful controller structures are for instance reduced-order controllers ($n_K \ll n_x$), decentralised or PID controllers. For PIDs the structure is

$$K_{\text{pid}}(\mathbf{x}) = \left[\begin{array}{cc|c} 0 & 0 & R_i \\ 0 & -\tau I_{m_2} & R_d \\ \hline I_{m_2} & I_{m_2} & D_K \end{array}\right] \tag{13}$$

where $\mathbf{x} = (\tau, \text{vec}(R_i), \text{vec}(R_d), \text{vec}(D_K))$ has $\dim(\mathbf{x}) = 3m_2p_2 + 1$, and a constraint $\tau \geq \epsilon$ (for some $\epsilon > 0$) is typically added in parameter space.

Armed with this, the following algorithm is proposed: The difference with Algorithm 1 in Fig. 1 is that LTR is no longer available to calibrate the procedure. Instead, the lower bound $r_*$ is computed in Step 2, based on a structured $H_\infty$-synthesis with objective $\mathcal{R}$. This can be obtained via the matlab function `hinfstruct` [18]. The mixed $H_2/H_\infty$-program is solved via [15], using the matlab function `fmincon` [19] as a presolver.

In order to solve (16) efficiently, the solution $\mathbf{x}^*$ of Step 1, or the solution $\mathbf{x}_\infty$ of Step 2, can be used as starting points. It is also possible to obtain a starting point $\mathbf{x}_r$ by stopping the minimisation in (15) at the moment when $\mathcal{R}(\mathbf{x}_r) \leq r$ is activated. This feature is indeed available in the matlab function `hinfstruct` [18]. The controller $K(\mathbf{x}_r)$ is then a favourable initial guess in (16), because it already satisfies the constraint. The result extending Proposition 1 is as the following:

*Proposition 4:* Suppose $\mathbf{x}_r$ with $\mathcal{R}(\mathbf{x}_r) = r$ is obtained as intermediate solution in Step 2 of Algorithm 2 in Fig. 4 and used as initial guess in solving program (16). Then the locally optimal solution $K(\mathbf{x}_{2,\infty}(r))$ of (16) is at least as robust as $K(\mathbf{x}_r)$ and has better $H_2$ performance.

*Proof:* The first statement says $\mathcal{R}(K(\mathbf{x}_{2,\infty}(r))) \leq r = \mathcal{P}(K(\mathbf{x}_r))$ which is clear, because a locally optimal solution is also feasible.

The second statement follows from the fact that $\mathbf{x}_r$ is used as initial guess. Then a descent method will produce a locally optimal solution, which has better performance than $K(\mathbf{x}_r)$. □

*Remark 9:* Note that solutions to (14), (15) and (16) may no longer be computed by algebraic Riccati equations or linear matrix inequalities (LMIs). While (14) can be solved by smooth optimisation technique, see for example, [20], programs (15) and (16) are non-smooth and require specific bundle techniques. (bilinear matrix inequality (BMI) solvers could at least in principle be used, but they suffer from the presence of Lyapunov variables, which lead to numerical trouble.) For non-smooth $H_\infty$ synthesis [21], and also [22–24], can be cited. A recent implementation is `hinfstruct` in [18], which is based on [21]. Constrained programs like (16) are discussed in [15, 25]. General mathematical background is given in [26, 27]. A recent approach to combine non-smooth techniques with classical non-linear programming techniques is discussed in [28].

*Remark 10:* In Algorithm 2 in Fig. 4 we assume that $P$ is stabilisable and detectable. However, we need to be able to stabilise the plant internally with a controller $K(\mathbf{x})$ of the imposed structure. Interestingly, deciding whether or not such controllers exists is nondeterministic polynomial (NP)-complete for many practical structures like PID, reduced-order, static, decentralised controller; see [29]. Practical ways to compute a stabilising $K(\mathbf{x})$ are discussed in [30].

## 8 Numerical experiments

In this section, we present three studies in which the proposed trade-off based on mixed structured $H_2/H_\infty$-control is tested. In each study performance of the nominal system is evaluated in the $H_2$-norm, which is optimised subject to a constraint on the controller structure (observer-based, respectively, PID). In the first and second study the input sensitivity function, $S$, and in the third the output sensitivity function, $\tilde{S}$, is used to assess robustness.

### 8.1 Mass-spring system

Our first study uses the mass-spring system [10] of Fig. 3, which can be considered as a prototype of a flexible system. Considering the positions and the velocities of the two mass as the states $x = [x_1\ x_2\ \dot{x}_1\ \dot{x}_2]^\top$, the state-space representation is

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k}{m_1} & \dfrac{k}{m_1} & -\dfrac{f}{m_1} & \dfrac{f}{m_1} \\ \dfrac{k}{m_2} & \dfrac{-k}{m_2} & \dfrac{f}{m_2} & \dfrac{-f}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix} u$$

$$y = [0\ 1\ 0\ 0]$$

*8.1.1 Linear quadratic Gaussian-loop transfer recovery:* According to Algorithm 1 in Fig. 1 the procedure starts with a nominal LQG synthesis. The nominal LQG controller $K_{\mathrm{LQG}}$ is obtained with the covariance matrices $W = BB^\top$ and $V = 1$, while $Q = C^\top C$ and $R = 1$; see [10]. This results in $(K_\mathrm{f}, K_\mathrm{c}) = ([0.94\ \ 0.06\ \ 0.97\ \ 0.75],\ [1.491.93\ \ 0.13\ \ 1.87])$ with performance $p^* = \mathcal{P}(K_{\mathrm{LQG}}) =$ 3.99. Following Algorithm 1 in Fig. 1, the LTR procedure is now applied to generate a curve $(K_\mathrm{f}(\rho), K_\mathrm{c})$. This is done by keeping $W$ fixed and letting $V = \rho I = \rho \to 0$, which corresponds to using the input sensitivity function $\mathcal{R}(K) := S(K) = (I + KG)^{-1}$ as robustness index. The LQG-LTR procedure is compared to our $H_2/H_\infty$ trade-off model (9) of Section 4. Fig. 2 compares performance and robustness of the different controllers. The graph $r(\rho)$ represents the robustness of both the LTR and the $H_2/H_\infty$ controller, which are matched through the constraint in program (9). As can be seen, performance is considerably improved without degrading robustness.

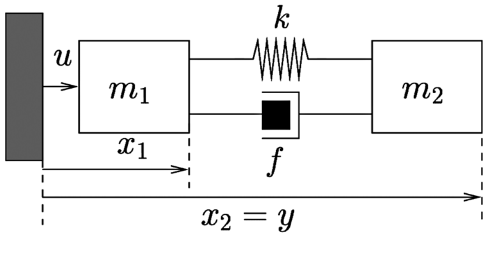


**Fig. 3** *Mass-spring system: nominal data are $m_1 = m_2 = 0.5$ kg, $k = 1$ N/m, $f = 0.0025$* Ns/m. *Measured output is $y = x_2$, control force u acts on $m_1$*

The parametric robustness of the LTR and the mixed $H_2/H_\infty$-controller have also been compared when mass $m_2$ and spring coefficient $k$ undergo changes around their nominal values, $k^0$ and $m_2^0$, in the square $(k^0 \pm 30\%k^0, m_2^0 \pm 30\%m_2^0)$. Fig. 5 compares the stability regions for $\rho = 0.001$. The performance $\mathcal{P}(K(0.001)) = 27.85$ of the LTR controller corresponds to a degradation of $\alpha = 597\%$ of the nominal performance $p^* = 3.99$. Image **(c)** shows what the mixed $H_2/H_\infty$ controller $K_{2,\infty}(\rho)$ achieves at the same $\rho = 0.001$. On top of having significantly better performance $\mathcal{P}(K_{2,\infty}(0.001)) = 4.23$, corresponding to $\alpha = 6\%$, it has also better parametric robustness. Fig. 6 displays the relative performance $(\mathcal{P}(G, K) - \mathcal{P}(G^0, K))/(\mathcal{P}(G^0, K))$ for

---

**Algorithm 2**

1: **Nominal synthesis.** Compute structured optimal $H_2$ controller $K(\mathbf{x}^*)$ by solving the nominal structured $H_2$ problem

$$\begin{aligned} &\text{minimise} \quad \mathcal{P}(\mathbf{x}) = \|T_{w_2 \to z_2}(P, K(\mathbf{x}))\|_2 \\ &\text{subject to} \quad K(\mathbf{x}) \text{ internally stabilising} \end{aligned} \qquad (14)$$

Evaluate its sensitivity $r^* = \mathcal{R}(K(\mathbf{x}^*)) = \|S(G, K(\mathbf{x}^*))\|_\infty$. If $r^*$ is small enough, meaning that $K(\mathbf{x}^*)$ is sufficiently robust, then quit. Otherwise continue and keep $r^*$ as upper bound.

2: **Lower bound.** Compute structured $H_\infty$-optimal controller $K(\mathbf{x}_\infty)$ by solving

$$\begin{aligned} &\text{minimise} \quad \mathcal{R}(\mathbf{x}) = \|S(G, K(\mathbf{x}))\|_\infty \\ &\text{subject to} \quad K(\mathbf{x}) \text{ internally stabilising} \end{aligned} \qquad (15)$$

Keep $r_* = \mathcal{R}(K(\mathbf{x}_\infty))$ as lower bound. Choose $r \in [r_*, r^*]$.

3: **Optimise.** For the current $r \in [r_*, r^*]$, solve the following structured mixed $H_2/H_\infty$ program

$$\begin{aligned} &\text{minimise} \quad \mathcal{P}(\mathbf{x}) = \|T_{w_2 \to z_2}(P, K(\mathbf{x}))\|_2 \\ &\text{subject to} \quad \mathcal{R}(\mathbf{x}) = \|S(G, K(\mathbf{x}))\|_\infty \le r \\ &\qquad\qquad K(\mathbf{x}) \text{ internally stabilising} \end{aligned} \qquad (16)$$

The locally optimal solution is $K(\mathbf{x}_{2,\infty}(r))$.

4: **Evaluate.** Check whether $K(\mathbf{x}_{2,\infty}(r))$ offers an acceptable compromise between performance and robustness. If it is not sufficiently robust, choose a smaller $r \in [r_*, r^*]$. If it is too robust and lacks performance, use larger $r \in [r_*, r^*]$. Then loop back to Step 3.

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**Fig. 4** *Algorithm 2: Trade-off between robustness and performance for structured $H_2$-synthesis*



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the controllers $K$ of Fig. 5 when the same variation of the nominal parameters is considered. Since LQG and LQG-LTR controllers are both not stabilising over the entire square, their graphs are restricted to their closed-loop stability regions. As can be seen, the mixed controller $K_{2,\infty}(\rho)$ performs best with regard to this criterion over the square.

*8.1.2 $H_2$-optimal PID controller:* In this section, a desensibilised $H_2$-optimal PID controller is searched for the mass-spring system. As LTR is no longer available, the procedure follows Algorithm 2 in Fig. 4, which starts by computing the solution of the nominal program (14) for the structure (13). The $H_2$-optimal PID controller $K_{\text{pid},2}$ has $p^* = \mathcal{P}(K_{\text{pid},2}) = 12.61$ and $r^* = \mathcal{R}(K_{\text{pid},2}) = 17.23$. Continuing with Algorithm 2 in Fig. 4, program (15) for the structure (13) is solved, which provides the most robust PID controller with regard to the sensitivity function $S$. This robustified PID has performance $p_* = \mathcal{P}(K_{\text{pid},\infty}) = 152.8$, which is clearly degraded ($p^* \ll p_*$), while naturally $r_* = \mathcal{R}(K_{\text{pid},\infty}) = 6.39$ is improved ($r_* < r^*$). Finally, the compromise is achieved by solving program (16), which it is initialised with $K_{\text{pid},\infty}$. Several choices $r \in [r_*, r^*]$ were tested, and finally $r = 17$ was chosen, because it achieved parametric robustness of $K_{\text{pid},2,\infty}$ over the 40% square of variation in $m_2, k$. Comparison with the two other PIDs is made in Fig. 7, where it can be seen that the mixed controller shows the best trade-off between performance and robustness (in the sense of the input sensitivity and parametric robustness).

### 8.2 Lateral flight control of an F-16 aircraft

In our last study, the improved LTR procedure was applied to lateral flight control of an F-16 aircraft. The non-linear F-16 lateral model was linearised using the F-16 simulation program [31]. The high-fidelity model is evaluated at altitude $h = 4575\,\text{m}$ and velocity $v = 152.5\,\text{m/s}$, considering steady wings-level flight conditions for trimming. The state variables are side slip angle $\beta$, bank angle $\phi$, roll rate $p$ and yaw rate $r$. Using a 6-DOF flat-earth, body-axis aircraft model

$$\begin{aligned}
\dot{\Phi} &= \frac{\cos\gamma_0}{\cos\theta_0} p_s + \frac{\sin\gamma_0}{\cos\theta_0} r_s \\
\dot{\beta} &= \frac{Y_\beta}{V}\beta + \frac{Yr}{V} r_s + \frac{g\cos\theta_0}{V}\Phi - r_s \\
\dot{p}_s &= L_\beta\beta + L_p p_s + L_r r_s + \delta_l(p_s, r_s) + L_{\delta_\text{a}}(\beta, \delta_\text{a}) + L_{\delta_\text{r}}(\beta, \delta_\text{r}) \\
\dot{r}_s &= N_\beta\beta + N_p p_s + N_r r_s + \delta_n(p_s, r_s) + N_{\delta_\text{a}}(\beta, \delta_\text{a}) + N_{\delta_\text{r}}(\beta, \delta_\text{r})
\end{aligned}$$

where

$$\begin{aligned}
p_s &= p\cos\alpha_0 + r\sin\alpha_0 \\
r_s &= r\cos\alpha_0 - p\sin\alpha_0
\end{aligned}$$

$\delta_\text{r}$ and $\delta_n$ are incremental rolling and yawing moment owing to $p_s$ and $q_s$. $L_{\delta_\text{a}}$, $L_{\delta_\text{r}}$, $N_{\delta_\text{a}}$ and $N_{\delta_\text{r}}$ are rolling and yaw moments because of aileron and rudder deflections. $\theta_0$, $\gamma_0$

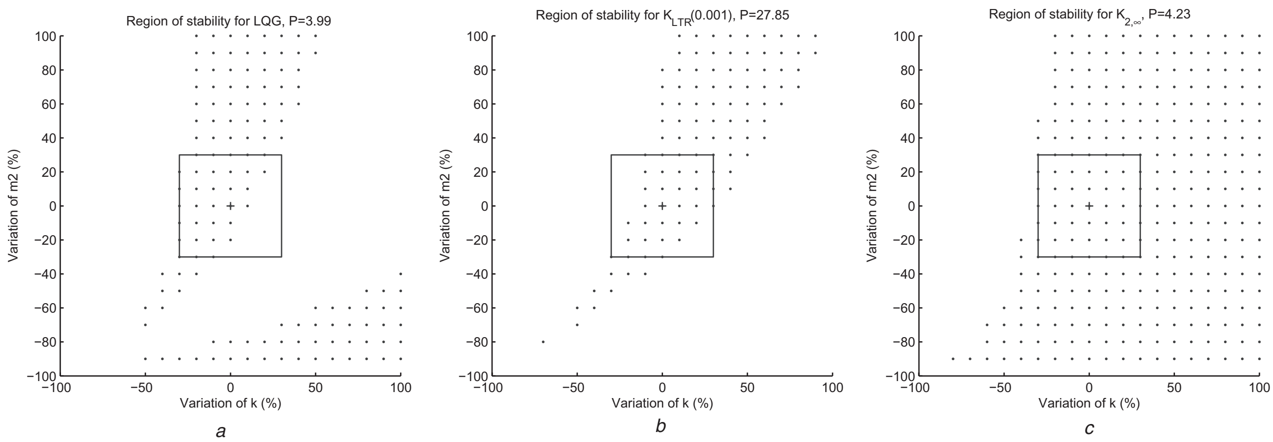


**Fig. 5** *LQG-LTR study: stability regions as a function of the parameters variation*

LQG controller has P = 3.99, which is too low. LQG/LTR controller shows enhanced robustness, but does not give parametric robustness. In addition, P = 27.85 is strongly degraded. Mixed $H_2/H_\infty$ controller is parametrically robust, and has better performance P = 4.23

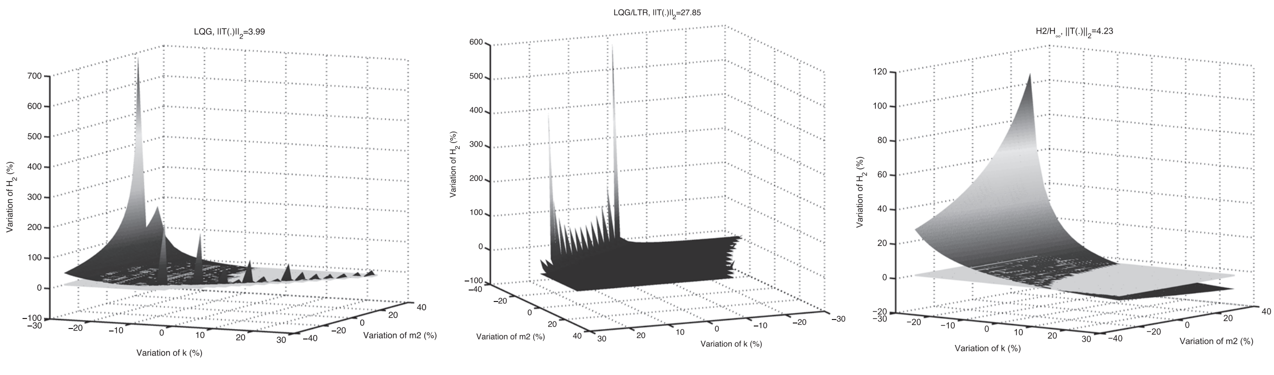


**Fig. 6** *LQG-LTR study: each graph shows relative performance as a function of the parameters variation*

Left: LQG controller, middle: LQG-LTR controller, right: mixed $H_2/H_\infty$


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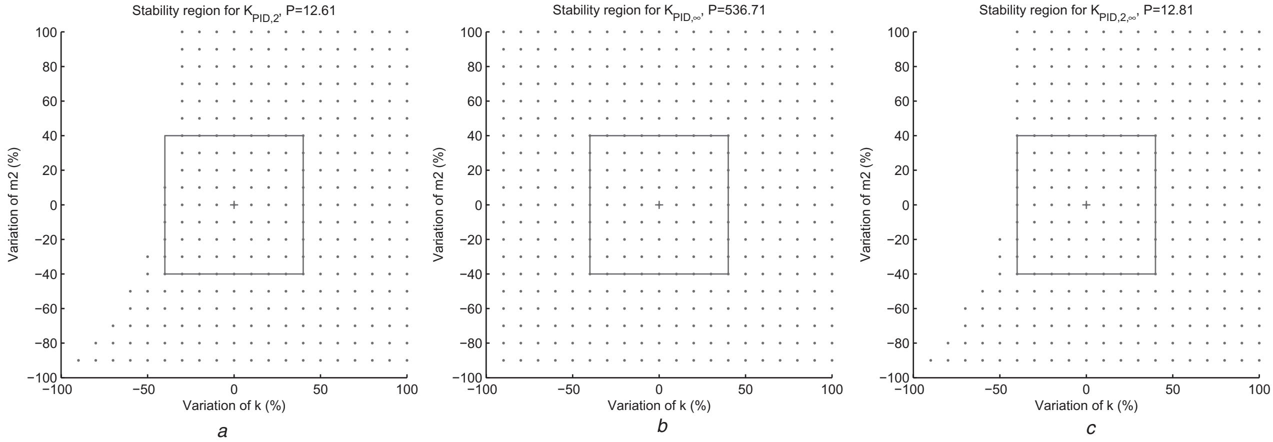


**Fig. 7** *PID study: stability regions as a function of the parameters variation*

Nominal controller (a) is not parametrically robust over the square. PID $H_\infty$-controller (b) is overly robust with degraded performance and provides lower bound. $H_2/H_\infty$-PID-controller (c) is parametrically robust and achieves good compromise between performance and robustness

and $\alpha_0$ are the trimmed pitch angle, angle of attack and side slip where $\alpha_0 = \theta_0 - \gamma_0$. For more details see [32, 33].

*8.2.1 Performance channel:* As in [34], state variables $\delta_a$ and $\delta_r$ representing deflection of aileron and rudder actuators are included in the model, each with approximate transfer function $20.2/(s + 20.2)$. The goal of the study is to make the bank angle $\phi$ follow a reference command $r_\phi$, while simultaneously keeping the side slip angle $\beta$ as close to $r_\beta = 0$ as possible. The plant has $u = [u_\phi\ u_\beta]$ as control input and $y = [\phi\ \beta]$ as measured output and is of type-0 with constant steady-state error. To eliminate this error, the dynamics are augmented by integrators in each control channel. Moreover, to balance the singular values at dc, the system was augmented again by the inverse of the dc gain of the system [34]. The overall state vector including aircraft state variables, actuators and integrators is then $x = [\beta, \phi, p, r, \delta_a, \delta_r, \epsilon_\phi, \epsilon_\beta]$. The model for synthesis is shown in Fig. 8, $G(s)$. In this figure the precompensator block represents the inverse of the dc gain. This figure also demonstrates the observer structure $K(s)$.

### 8.3 LTR procedure

In this study, LTR recovery at the output breaking point is used, that is, robustness is measured via the output sensitivity function $\tilde{S}$, and an observer-based controller is computed. Using $V = I_2$ and $W = \text{diag}$

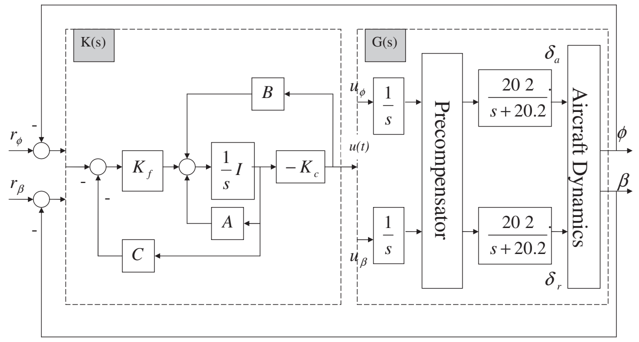


**Fig. 8** *Model of F16 aircraft lateral control system and the observer structure*

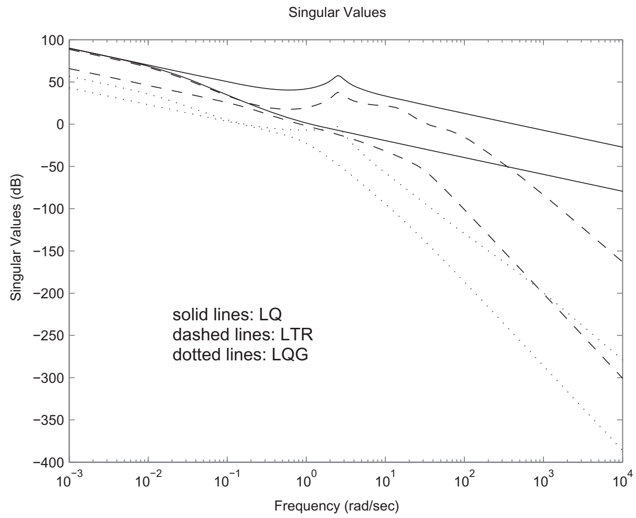


**Fig. 9** *Singular values of the loop transfer function $L(s) = G(s)K(s)$ for LQ, LQG and LTR controller*

([0.1 0.1 0.1 0.1 0 0 10 10]) $\times$ 100, we first fix the Kalman gain $K_f$ such that the target loop gain $C(sI - A)^{-1}K_f$ has the desired performance. That this goal is achieved can be seen in the singular value plot in Fig. 9, and through the step responses of Fig. 10 (solid lines).

LTR is now applied with $Q = C^\top C$ and $R = \rho I_2$, where $\rho \to 0$, and $K_c(\rho)$ is tuned. With $q = 1/\rho$ this corresponds to the case discussed in Section 5. Fig. 9 compares the singular values $\overline{\sigma}$ and $\underline{\sigma}$ of the loop transfer functions of LQ (target) with those of LQG and LTR($\rho = 1e - 10$). In other words, the singular values of $C(sI - A)^{-1}K_f$ are compared with the singular values of $G(s)K_c(\rho)(sI - (A + BK_c(\rho) + K_f^\top C))^{-1}K_f$ for $\rho = 1$ and $\rho = 1e - 10$. As can be seen, forcing $\rho \to 0$ brings the singular values of the LTR controller near those of the target. In addition, this also drives the system output responses towards the model responses of the target, as shown in Figs. 10*a* and *b*. In Figs. 10*c* and *d*, the control input signals of LQ, LQG and LTR are compared.

Unfortunately, the LTR controller causes a large control input, which results in a large (degraded) performance. This loss of performance increases with $\rho^{-1}$ as Fig. 11 shows. In the same figure the robustness index $\mathcal{R}_{LTR} =$



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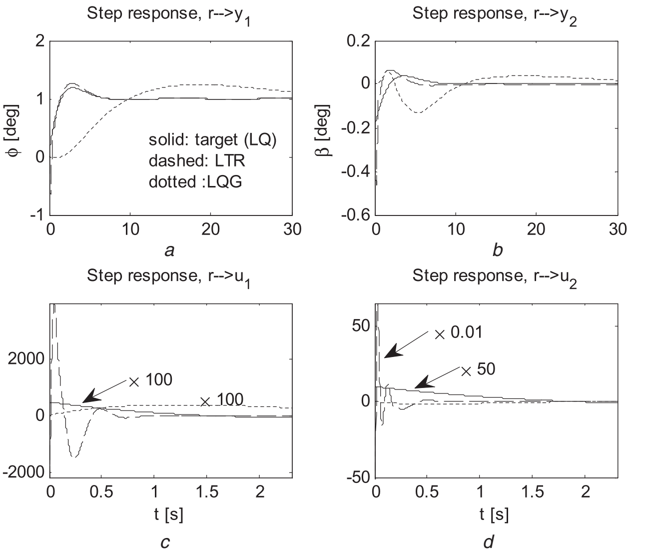


**Fig. 10** *Step responses with different controllers*

Step responses of LTR and of $H_2/H_\infty$ controllers a: controlled output, b: command input

$\|\tilde{S}\|_\infty = \|(I + GK_{\text{LTR}}(\rho))^{-1})\|_\infty$ is displayed. As can be seen, at the beginning (going from right to left) $\mathcal{R}$ increases and then decreases before stabilising around $\mathcal{R}_{\text{LQ}} = \|(I + GK_{\text{LQ}}(\rho))^{-1})\|_\infty$. This proves that LTR with recovery at the output breaking point is not a monotone procedure either.

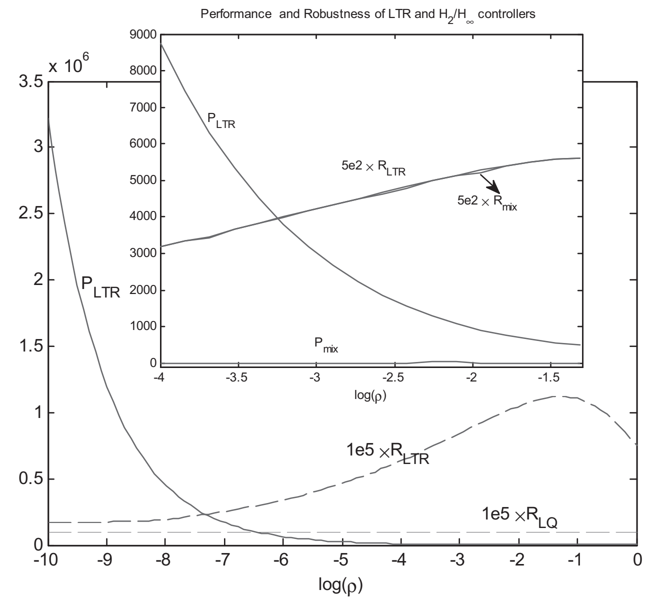


**Fig. 11** *F-16 study: comparison of performance of LTR and $H_2/H_\infty$ controller when robustness according to $\mathcal{R}(K) = \|\tilde{S}(G, K)\|_\infty$ is used*

*8.3.1 Mixed synthesis:* In order to overcome the loss of performance of the LTR controller, we apply Algorithm 1 in Fig. 1, where in program (16) the output

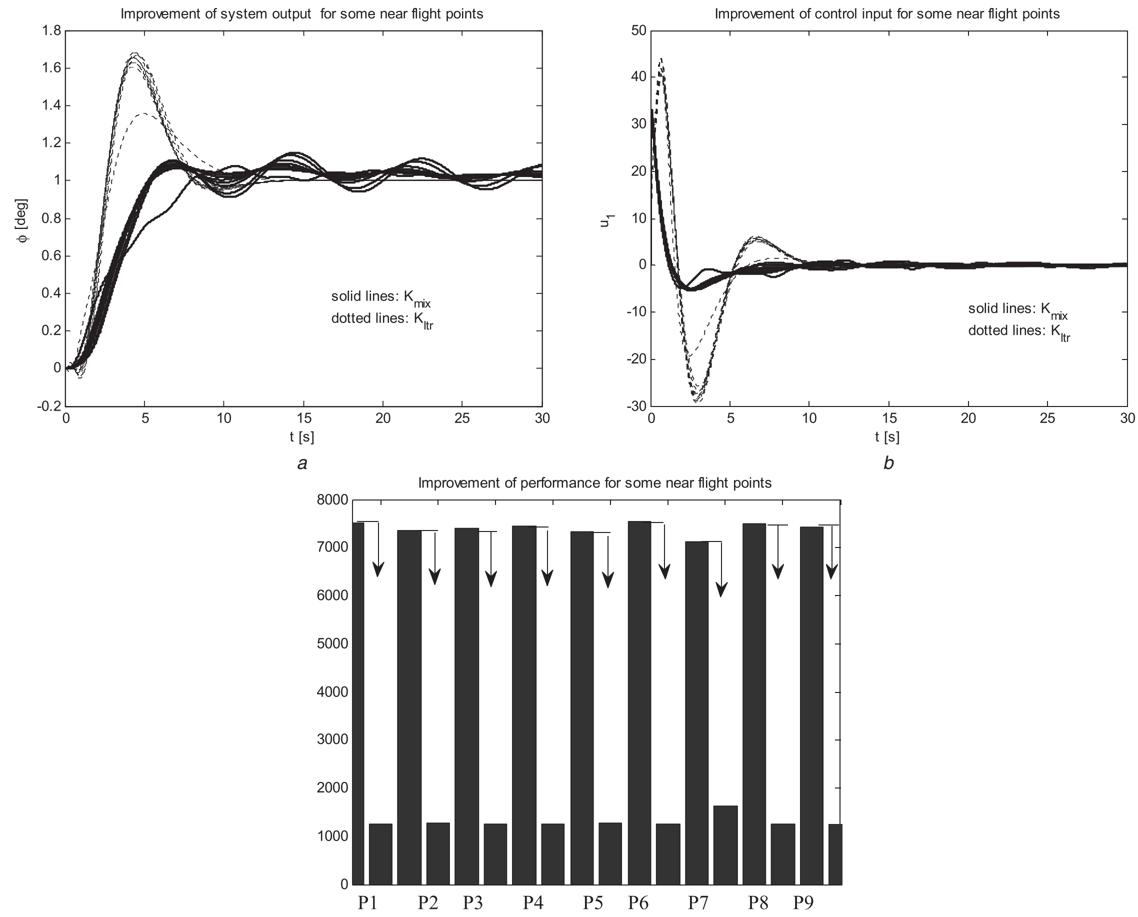


**Fig. 12** *Comparing LTR and $H_2/H_\infty$ controllers against the system variations*

*a* and *b* Step responses

*c* Performance


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sensitivity function $\tilde{S}$ replaces $S$. An appropriate parameter range is $\rho \in [10^{-4}\ 10^{-1.3}]$, where robustness $\mathcal{R}$ decreases monotonically with $\rho$, while performance $\mathcal{P}$ increases. Fig. 11 compares performance after matching robustness of the $H_2/H_\infty$ and LTR controllers via Proposition 1. A substantial improvement in performance can be observed.

The efficiency of this new method is checked by considering changes of the flight parameters. $h = h_0 \pm \Delta h$ and $v = v_0 \pm \Delta v$ are considered with $\Delta h = 305\,\text{m}$ and $\Delta v = 7.625\,\text{m/s}$, the nominal flight point being $h_0 = 4575\,\text{m}$ and $v_0 = 152.5\,\text{m/s}$. The LTR controller and the corresponding mixed controller are evaluated at $\rho = 1.438e-4$. Figs. 12*a* and *b* compares the first output and the first control input of the eight neighbouring flight points around the nominal flight point. The diagram in Fig. 12*c* shows the improvement in performance obtained with the mixed controller.

## 9 Conclusion

We have used mixed $H_2/H_\infty$ synthesis with structured control laws to obtain a quantified trade-off between performance and robustness. Within the class of observer-based controllers our method leads to an improvement of the LQG-LTR procedure. The latter is still useful to calibrate and initialise the procedure. For other controller structures a different idea is used to calibrate the mixed program. The new method was applied to a mass-spring benchmark example and also to lateral flight control of an F-16 aircraft. Experiments indicate that the new technique can also be useful to enhance the parametric robustness of a design. In our tests the achieved degree of parametric robustness was satisfactory.

## 10 Acknowledgment

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