SIAM J. CONTROL OPTIM.
Vol. 38, No. 4, pp. 1223–1240

ADAPTIVE IMAGE RECONSTRUCTION USING INFORMATION MEASURES*

ULRIKE HERMANN† AND DOMINIKUS NOLL‡

Abstract. We present a class of nonlinear adaptive image restoration filters which may be steered to preserve sharp edges and contrasts in the restorations. From a theoretical point of view we discuss the associated variational problems and prove existence of solutions in certain Sobolev spaces $W^{1,p}$ or in a BV-space. The degree of regularity of the solution may be understood as a mathematical explanation of the heuristic properties of the designed filters.

Key words. adaptive image restoration filter, image restoration, variational methods

AMS subject classifications. 49J52, 49N15, 94A17

PII. S0363012997324338

1. Motivation and purpose.

Various inverse problems require reconstructing an unknown density function $u(x)$, $x \in \Omega \subset \mathbf{R}^n$, from a finite number of measurements of the form

$$\int_\Omega \big(a_k(x)u(x) + b_k(x) \cdot \nabla u(x)\big)\, dx \;=\; c_k, \qquad k = 1, \ldots, N. \tag{1.1}$$

Examples of particular interest are in medical imaging, where the data c_k represent attenuation coefficients of transmission x-rays, or in image restoration, where the c_k are gray levels at pixels k of a blurred version of the true image $u(x)$. Restoring the original $u(x)$ is usually an ill-posed problem, and the inevitable measurement noise may make this a difficult task. One way to restore $u(x)$ in the presence of noise is to stabilize inversion of (1.1) by introducing a regularizing functional of the form

$$I[u] \;=\; \int_\Omega h\big(u(x), \nabla u(x)\big)\, dx, \tag{1.2}$$

closely related to the specific restoration problem. Introducing linear operators A, B by

$$(Au)_k = \int_\Omega a_k(x)u(x)\, dx, \qquad (Bv)_k = \int_\Omega b_k(x) \cdot v(x)\, dx, \tag{1.3}$$

we consider the following inverse methods which we call the *tolerance* and the *penalization approaches,* respectively:

$$(P)_{\text{tol}} \qquad \begin{array}{ll} \text{minimize} & I[u] \\ \text{subject to} & \big|Au + B\nabla u - c\big| \le \varepsilon, \\ & \displaystyle\int_\Omega u(x)\, dx = 1 \end{array}$$

*Received by the editors July 11, 1997; accepted for publication (in revised form) June 1, 1999; published electronically April 20, 2000.
http://www.siam.org/journals/sicon/38-4/32433.html
†Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany.
‡Université Paul Sabatier, Mathématiques pour l'Industrie et la Physique, 118, route de Narbonne, 31062 Toulouse, France (noll@dumbo@ups-tlse.fr).

and

$$(P)_{\text{pen}} \qquad \begin{array}{ll} \text{minimize} & I[u] + \frac{\alpha}{2}|Au + B\nabla u - c|^2 \\ \text{subject to} & \displaystyle\int_\Omega u(x)\,dx = 1 \end{array}$$

($|\cdot|$ = Euclidean norm). A well-known method based on the scheme $(P)_{\text{pen}}$ is *Tychonov-regularization*, where the functional (1.2) is a square norm of $u(x)$ or $\nabla u(x)$ (cf. [16, 12, 13]). However, in image restoration this choice is known to produce poor results, and more sophisticated functionals $I[u]$ are required. In our present work, we shall consider a class of functionals $I[u]$ of information type that are particularly suited for image restoration problems and that we motivate by a heuristic argument. The remaining parts of the paper address the mathematical problems arising from this choice.

The values $u(x)$ are relative gray levels of the unknown image, hence the normalization $\int u\,dx = 1$. Since gray levels are nonnegative, we require reconstructions $u(x) \geq 0$, and this may be guaranteed by our choice of $I[u]$. For the moment consider the model $(P)_{\text{tol}}$. The data being noisy, we should not force equality $Au + B\nabla u = c$, but allow for a tolerance $\varepsilon > 0$, typically estimated using a χ^2-statistics (cf. [18]). The role of the functional $I[u]$ is now to avoid picking highly irregular objects u which would fit the tolerance condition. In other terms, minimizing $I[u]$ subject to the constraint $|Au + B\nabla u - c| \leq \varepsilon$ to some degree means filtering the unknown object $u(x)$. However, as mentioned before, default choices like $I[u] = \int_\Omega |\nabla u|^2\,dx$ tend to smooth away sharp edges in the image. Smoothing while retaining edges is needed, and this requires adapting the filter to the image.

Consider the class of functionals (1.2) defined through the integrands

$$h(u,\xi) = \begin{cases} u\phi\big(\xi/u\big) & \text{if } u > 0, \\ \phi^{0+}(\xi) & \text{if } u = 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

where $\phi : \mathbf{R}^n \to \mathbf{R}$ is a convex functional and ϕ^{0+} denotes its recession function, needed to render the functional h lower semicontinuous (lsc),

$$\phi^{0+}(\xi) = \sup_{t>0} \frac{\phi(\eta + t\xi) - \phi(\eta)}{t},$$

for an arbitrary fixed η in domϕ (cf. [21, p. 66ff]). Then h is jointly convex in (u, ξ), and (1.4) will be called *Csiszár information measures*. An important special case is $\phi(t) = |t|^2$, which is Fisher's information (cf. [19]). Notice that since $h(u, \xi) = +\infty$ for $u < 0$, the objectives (1.4) force nonnegative solutions, as required.

In order to motivate the inverse approach based on (1.4), let us specialize even further by considering functionals of the form $\phi(\xi) = \psi(|\xi|)$ for convex $\psi : \mathbb{R} \to \mathbb{R}$. Since $|\nabla u|$ is invariant under rigid motions, so is $h(u, \nabla u)$ defined through (1.4); hence, this choice will lead to methods invariant under rigid motions of the image. Proceeding in a purely formal way, we first do a change of variables $u(x) = e^{v(x)}$ to account for the condition $u(x) > 0$. The Euler–Lagrange equation for the transformed problem $(P)_{\text{pen}}$ is then

$$\begin{aligned} -\operatorname{div}\left(\frac{\psi'\big(|\nabla v|\big)}{|\nabla v|}\nabla v\right) - \psi'\big(|\nabla v|\big)|\nabla v| + \psi(|\nabla v|) \\ + \alpha(A^T - \operatorname{div}B^T)(Ae^v + B(\nabla e^v) - c) \;=\; 0 \end{aligned} \tag{1.5}$$

with adjoints A^T, B^T, and div B^T defined as

$$A^T(\lambda) = \sum_{k=1}^{N} \lambda_k a_k, \qquad B^T(\lambda) = \sum_{k=1}^{N} \lambda_k b_k, \qquad \mathrm{div} B^T(\lambda) = \sum_{k=1}^{N} \lambda_k \mathrm{div} b_k.$$

Consider the case $\Omega \subset \mathbb{R}^2$. Following an idea originating from [8] and extended in [2, 26], sharp edges (contrasts) in the image $v(x, y)$ occur along level curves $v(x, y) = c$, the indication being that the gradient $\nabla v(x, y)$ becomes large. In this case, smoothing across the edge $v(x, y) = c$ should be dispensed with, while smoothing along the edge is still needed to suppress irregular behavior.

For a point (x, y) on the level curve $v = c$, consider the adapted cartesian coordinates $T(x, y), N(x, y)$ meaning tangential and normal directions to the level curve $v = c$ at (x, y):

$$N(x, y) = \frac{\nabla v(x, y)}{|\nabla v(x, y)|}, \qquad T(x, y) \perp N(x, y).$$

Expanding the divergence term in (1.5) gives

$$-\mathrm{div}\left(\frac{\psi'\big(|\nabla v|\big)}{|\nabla v|}\,\nabla v\right) \;=\; -\frac{\psi'(|\nabla v|)}{|\nabla v|}\,\Delta v \;-\; \left(\frac{\psi''(|\nabla v|)}{|\nabla v|^2} \;-\; \frac{\psi'(|\nabla v|)}{|\nabla v|^3}\right)\nabla v \cdot \nabla^2 v \cdot \nabla v.$$

Observe that the Laplacian is invariant under orthogonal transformations, $\Delta v = v_{xx} + v_{yy} = v_{TT} + v_{NN}$, and secondly that $\frac{\nabla v}{|\nabla v|} \cdot \nabla^2 v \cdot \frac{\nabla v}{|\nabla v|} = v_{NN}$. Then the Euler equation in (T, N)-coordinates reads

$$\begin{aligned} -\left(\frac{\psi'(|\nabla v|)}{|\nabla v|}\right) v_{TT} \;-\; \psi''(|\nabla v|)\, v_{NN} \;-\; \psi'(|\nabla v|)|\nabla v| \;+\; \psi(|\nabla v|)& \\ + \alpha(A^T - \mathrm{div}B^T)(Ae^v + B(\nabla e^v) - c) &= 0. \end{aligned}$$

Suppose $|\nabla v|$ is small, indicating that $v = c$ is not an edge, and hence smoothing should be encouraged. Assuming (i) $\psi'(0) = 0$ and (ii) $\psi''(0) > 0$, in a neighborhood of (x, y), the Euler equation is qualitatively of the form

$$-\psi''(0)\big(v_{TT} + v_{NN}\big) \;+ \psi(0) \;+\; \alpha(A^T - \mathrm{div}B^T)(Ae^v + B(\nabla e^v) - c) = 0.$$

Due to $v_{TT} + v_{NN} = v_{xx} + v_{yy} = \Delta v$, this may be considered as having a strong smoothing effect around (x, y).

Assume, on the other hand, that $|\nabla v|$ is large at (x, y), indicating an edge. Then we wish to smooth in T-direction but not in N-direction. This is achieved, for instance, by having

$$\text{(iii)} \qquad \frac{\psi'(t)}{t} \gg \psi''(t)$$

for large t. The coefficient of v_{NN} then being negligible in a neighborhood of (x, y), the differential equation is qualitatively of the form

$$-C\, v_{TT} - \psi'(|\nabla v|)|\nabla v| + \psi(|\nabla v|) + \alpha(A^T - \mathrm{div}B^T)(Ae^v + B(\nabla e^v) - c) = 0,$$

indicating a preference for smoothing in T-direction, since as before the tendency to smoothing is governed by the second order terms. As an example for (iii), consider a

ψ which for large t behaves like $\psi(t) = t^p$ for some $p > 1$. This gives $\frac{\psi'(t)}{t}/\psi''(t) = 1/(p-1)$, which could be made as large as desired by choosing p close to 1.

The conditions (i)–(iii) do not entirely fix the function ψ, so further evidence (theoretical and numerical) is needed to propose a best choice. The present paper, rather, addresses the theoretical aspects of the models $(P)_{\rm tol}$ and $(P)_{\rm pen}$, particularly the question of existence of solutions. Our method of proving existence may be considered a fairly general scheme including a large variety of possible applications. It does not rely on compactness arguments but exploits the convexity of the problems.

A second problem associated with the variational methods $(P)_{\rm pen}$ and $(P)_{\rm tol}$ is to justify the Euler–Lagrange equation (1.5), formally derived above. This problem, which is difficult, is treated in [10].

It is intuitively clear that the choices $\psi(t) = |t|^p$ discussed above should lead to image restorations exhibiting more and more sharp edges when $p > 1$ approaches 1. One way to corroborate this in the variational context is by showing that the solutions of the corresponding programs $(P)_{\rm pen}$ and $(P)_{\rm tol}$ are in a Sobolev space $W^{1,1+\varepsilon(p)}$, with $\varepsilon(p) \to 0$ as $p \to 1$ (cf. Example 3 at the end of section 6). In the limiting case $p = 1$, we would get solutions which degrade to BV-functions, allowing even for discontinuities. We mention that the latter is sometimes considered as a natural setting for image processing, particularly if the purpose is segmentation or edge detection (cf. [17, 5, 24, 27]).

Numerical experiments for special choices $\psi(t)$ have been presented in [19, 20]. The authors of [2, 26] report experiments with objectives of the form $h(u,\xi) = \phi(|\xi|)$ built on a related philosophy. A comparative study of adaptive filters will be presented elsewhere. We mention that the class of functionals (1.4) has various other applications. See in particular [18] for variational problems involving Fisher's information ($p = 2$).

2. Outline of the method. We start by giving an outline of our method of proving existence of solutions and then point to the steps which cause particular difficulties. Our approach may be called a *bidual relaxation* scheme: Writing (P) for any of the formulations $(P)_{\rm pen}$ or $(P)_{\rm tol}$ and proceeding in a formal way, we first obtain a concave dual program (P^*). Formal means that we do not try to find a dual pair of Banach spaces in which the duality may be justified rigorously. In a second step we repeat the same for (P^*), but this time we use the full convexity machinery. This means we prove a Lagrange multiplier theorem for (P^*). The multiplier $\bar{u}$ is an element of the dual Banach space $M(\bar{\Omega})$ of signed Radon measures and an optimal solution to a properly defined convex bidual (P^{**}). We may therefore interpret $\bar{u}$ as a generalized solution to the original program (P). In a third step we show that under mild additional conditions, we get a solution $\bar{u}$ in a Sobolev space or even in a classical space $C^1(\bar{\Omega})$.

Notice that this scheme has been used various times. However, the difficulties are in the details; in particular, technical problems arise if we are not satisfied with solutions in a BV-space, but wish to prove regularity results (cf. section 4). For complementary literature we refer to [7, 3, 4, 2].

Let us now consider some of the details. First, dualizing (P^*) requires a Lagrange multiplier theorem. This type of result typically needs a constraint qualification hypothesis, which should not be artificial in the light of the original problem (P). The existence result Proposition 4.1 in fact avoids any such hypothesis by providing a solution $u \in M(\bar{\Omega})$, the space of Radon measures.

The second step in our scheme is to show that the generalized solution $u \in M(\bar{\Omega})$

is a BV-function. This is done in Proposition 4.3 and requires a richness hypothesis $(A1)$. Condition $(A1)$ excludes objectives (1.2) where $h(x, \xi)$ is linear in ξ. With $h(x, \xi) = g(x) + \eta \cdot \xi$ linear in ξ, it is possible to construct examples where the generalized solution $\bar{u}$ is not a BV-function, although this may be guaranteed under coercivity assumptions on $g(x)$. We consider objectives (1.2) linear in ξ as of minor importance for possible applications and therefore do not pursue their analysis here.

In a third step of Proposition 4.6, we show that the solution $\bar{u}$, so far a BV-function, is an element of the Sobolev space $W^{1,1}(\Omega)$ if a slightly stronger regularity hypothesis $(A2)$ is satisfied. Hypothesis $(A2)$ may be understood as a weak coercivity condition on h, implying in particular that for fixed x, $h(x, \xi)$ grows stronger than linearly in ξ as $|\xi| \to \infty$.

In practice, it is often enough to have solutions in $W^{1,1}(\Omega)$, in particular, if the natural domain of the functional I_h is a better Sobolev space $W^{1,p}(\Omega)$ for some $p > 1$. Here the solution will automatically be an element of $W^{1,p}(\Omega)$. In section 6 we present an extended version of this observation, showing that under a stronger hypothesis $(A3)$, the solution $\bar{u}$ is improved to be of class $W^{1,p}(\Omega)$ for some $p > 1$, with the possibility to having classical solutions if p is large enough. Hypothesis $(A3)$ is seen to be a coercivity condition on h, satisfied, e.g., if h^* grows at most polynomially (see section 6).

We mention that bidual relaxation as presented here is not aimed at image restoration exclusively. In fact, the hypotheses $(A1)$–$(A3)$ are fairly general and ensure a broad applicability. Nonetheless, in image enhancement, $(A2)$ and $(A3)$ might be considered too strong, in particular under the agreement that images be best represented as BV-functions. This point of view, initiated by Osher and Rudin [23, 24, 25], is widely accepted if the aim is edge detection or segmentation (cf. [13, 15, 6]), although it is clear that many images continue to be modeled as continuous or even smooth functions. This is particularly so in cases where the physical image generating process is taken into account (astronomy, medical imaging). We hold that our approach of modeling images in Sobolev spaces may offer a compromise.

3. Lagrangian formulation for $(P)_{\mathrm{pen}}$. In this section we present the first part of the scheme for program $(P)_{\mathrm{pen}}$. We provide a suitable Lagrangian formulation and a corresponding concave dual program (P^*). The second step of the relaxation scheme, dualizing the dual to obtain the bidual, will be presented in section 4.

For the following, let us fix some notations and definitions. Let Ω be a bounded open subset of $\mathbb{R}^N$, and suppose $a_k \in C(\overline{\Omega})$, $b_k \in C^1(\overline{\Omega})^n$ for $k = 1, \dots, N$. (It would be sufficient to require piecewise continuity of a_k and piecewise continuous differentiability of b_k.) Then the linear operators A and B defined by (1.3) are bounded on $L_1(\Omega)$ and $L_1(\Omega)^n$, respectively.

Let $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper convex lsc function with nonempty domain $\operatorname{dom} h$ (cf. [21]). Then

$$I_h[u] = \int_\Omega h(u(x), \nabla u(x))\, dx$$

is a proper convex lsc functional defined for all $u \in W^{1,1}(\Omega)$. Notice that we do not exclude the possibility $I_h[u] = +\infty$, as would, for instance, occur for a classical functional like $\int_\Omega |\nabla u|^2\, dx$, whose natural domain is $W^{1,2}(\Omega)$. A value $I_h[u] = +\infty$ simply means that u does not contribute to the minimization process. On the other hand, $I_h[u] = -\infty$ is impossible as a consequence of the lower semicontinuity of I_h (cf. [22] for this and other facts about convex integral functionals).

In cases when we wish to force positivity of the solution, we require $h(u,\xi) = +\infty$ whenever $u < 0$. Then $I_h[u] = +\infty$ unless $u \geq 0$ almost everywhere (a.e.) on Ω.

To avoid trivial situations, we will generally assume that $(P)_{\text{pen}}$ and $(P)_{\text{tol}}$ are feasible. More precisely, we assume existence of a function $u \in C^1(\overline{\Omega})$ with $I_h[u] < \infty$, $\int_\Omega u(x)\,dx = 1$, respectively, $|Au + B\nabla u - c| \leq \varepsilon$ in the case of the tolerance model. The value of program (P) will be denoted by $V(P)$, and we will require $V(P) > -\infty$ because otherwise no optimal solution exists. So altogether we adopt $-\infty < V(P) < \infty$ as our standing hypothesis. In the present section we consider $(P)_{\text{pen}}$. Analogous results for $(P)_{\text{tol}}$ will be presented in section 5.

We proceed to give a Lagrangian formulation of $(P)_{\text{pen}}$. By introducing dummy variables $v = \nabla u$ and $e = Au + Bv - c$ and by defining

$$J_h(u,v) := \int_\Omega h(u(x), v(x))\,dx$$

we rewrite $(P)_{\text{pen}}$ in the form

$$(P)_{\text{pen}} \qquad \begin{aligned} &\text{minimize} && J_h(u,v) + \frac{\alpha}{2}|e|^2 \\ &\text{subject to} && \nabla u = v,\ Au + Bv - c = e, \\ &&& \int_\Omega u(x)\,dx = 1 \end{aligned}$$

with $e \in \mathbb{R}^N$ and $u \in C^1(\overline{\Omega}), v \in C(\overline{\Omega})^n$. This suggests using the Lagrangian

$$\begin{aligned} L(u,v,e;w,\lambda,\mu) = J_h(u,v) + \frac{\alpha}{2}|e|^2 + \langle w, \nabla u - v\rangle \\ + \lambda \cdot (Au + Bv - c - e) + \mu\left(\int_\Omega u(x)\,dx - 1\right), \end{aligned}$$

where $\langle .,. \rangle$ denotes the dual form either between $C(\bar{\Omega})^n$ and $M(\bar{\Omega})^n$ or between $L_1(\Omega)^n$ and $L_\infty(\Omega)^n$. We can now write $(P)_{\text{pen}}$ in the equivalent form

$$\inf_{u,v,e} \sup_{w,\lambda,\mu} L(u,v,e;w,\lambda,\mu). \tag{3.1}$$

As usual, the corresponding concave dual program is then defined by switching the inf and sup:

$$\sup_{w,\lambda,\mu} \inf_{u,v,e} L(u,v,e;w,\lambda,\mu). \tag{3.2}$$

We do not attempt to prove directly that (P) and (P^*) are equivalent or at least have equal values, since this will follow later as a consequence of the bidual relaxation scheme. Instead, we investigate (3.2) a little further by explicitly calculating the inner infimum.

To do this, we start by calculating the partial Legendre–Fenchel transform of L in its first three variables, defined as

$$L^*(y,z,d;w,\lambda,\mu) = \sup_{u,v,e} \big(\langle u, y\rangle + \langle v, z\rangle + e \cdot d - L(u,v,e;w,\lambda,\mu)\big),$$

and then recognize $-L^*(0,0,0;w,\lambda,\mu)$ as the objective of the dual (3.2), to be maximized over (w,λ,μ). While [21] is the basic reference for notions from finite dimensional convexity, a rigorous justification of (P^*) as obtained below would call for methods as used in section 4 or in [7]. In particular, it would require calculating the conjugate L^* with respect to the space $C(\bar{\Omega})$ and its dual $M(\bar{\Omega})$, the space of signed Radon measures on $\bar{\Omega}$. Instead of calculating $(J_h)^*$, defined on a space of measures, we restrict the dual to the classical spaces $C(\bar{\Omega})$ and $C^1(\bar{\Omega})^n$, where it suffices to calculate J_{h^*} with the approach described as formal in section 2.

Written as a convex program, the dual is of the following form:

$$(P^*)\qquad \begin{aligned} \text{minimize} \quad & J_{h^*}(y,z) + \frac{1}{2\alpha}|\lambda|^2 + \lambda \cdot c + \mu \\ \text{subject to} \quad & y = \operatorname{div} z + \operatorname{div} B^T\lambda - A^T\lambda - \mu, \\ & y \in C(\overline{\Omega}), z \in C^1(\overline{\Omega})^n, \lambda \in \mathbb{R}^N, \mu \in \mathbb{R}. \end{aligned}$$

Here h^* is the Legendre–Fenchel conjugate of h and $J_{h^*}(y,z) := \int_\Omega h^*(y(x), z(x))\,dx$.

Example 1. For the class of Csiszár information measures (1.4) we have

$$h^*(y,z) = \begin{cases} 0, & y + \phi^*(z) \leq 0, \\ \infty, & y + \phi^*(z) > 0. \end{cases}$$

As is easy to see, (P^*) has feasible points, so the value $V(P^*) < +\infty$. Also, the fact that the dual was obtained by flipping sup and inf gives $V(P^*) \geq -V(P) > -\infty$, so $V(P^*)$ is finite. The relation $V(P^*) \geq -V(P)$ is often referred to as weak duality (cf. [7]).

4. Existence of solutions for $(P)_{\text{pen}}$. The second part of our scheme now requires dualizing (P^*) again to obtain what we call a bidual relaxation (P^{**}) of the original program $(P)_{\text{pen}}$. As opposed to the formal way we employed to derive (P^*), we shall now have to rigorously dualize (P^*). As a consequence, the bidual (P^{**}) will be formulated in a dual Banach space, a space of measures. In a third step, also presented in the section, we will show that under reasonable conditions, the generalized solutions are functions in the Sobolev space $W^{1,1}(\Omega)$. A fourth step, to be presented in section 6, will examine under what circumstances a classical solution in $C^1(\overline{\Omega})$ may be obtained.

As before, duality requires an appropriate Lagrangian formulation, which we obtain by attaching a multiplier $u \in M(\bar{\Omega})$ to the equality constraint in (P^*). The dual Lagrangian is then

$$L_D(y,z,\lambda,\mu;u) = J_{h^*}(y,z) + \frac{1}{2\alpha}|\lambda|^2 + \lambda \cdot c + \mu + \langle u, \operatorname{div} z + \operatorname{div} B^T\lambda - A^T\lambda - \mu - y\rangle,$$

and an equivalent way of writing (P^*) is the minimax form:

$$(P^*)\qquad \inf_{\substack{y \in C(\overline{\Omega}) \\ z \in C^1(\overline{\Omega})^n \\ \lambda \in \mathbb{R}^N,\, \mu \in \mathbb{R}}} \ \sup_{u \in M(\overline{\Omega})} L_D(y,z,\lambda,\mu;u).$$

Switching inf and sup leads to the corresponding bidual program,

$$(P^{**})\qquad \sup_u \inf_{y,z,\lambda,\mu} L_D(y,z,\lambda,\mu;u),$$

and immediately gives $V(P^{**}) \leq V(P^*)$ (weak duality). Proving equality requires more work.

PROPOSITION 4.1. *We have $V(P^{**}) = V(P^*)$, and (P^{**}) admits an optimal solution $\overline{u} \in M(\overline{\Omega})$.*

Proof. (a) Let us first consider the case where h is not affine. The function $f : C(\overline{\Omega}) \to \mathbb{R} \cup \{+\infty\}$, defined by

$$f(\eta) = \inf \Bigg\{ J_{h^*}(y,z) + \frac{1}{2\alpha}|\lambda|^2 + \lambda \cdot c + \mu : y \in C(\overline{\Omega}), z \in C^1(\overline{\Omega})^n, \lambda \in \mathbb{R}^N, \mu \in \mathbb{R}, \\ \operatorname{div} z + \operatorname{div} B^T\lambda - A^T\lambda - \mu - y = \eta \Bigg\},$$

is proper convex lsc and we have $f(0) = V(P^*)$, which is finite. According to the general theory (cf. [7]) it remains to show that $\partial f(0)$, the subdifferential of f at 0, is nonempty, since every $\overline{u}$ with $-\overline{u} \in \partial f(0)$ is a solution to (P^{**}), showing in addition $V(P^*) = V(P^{**})$. Notice here that f, being defined on $C(\overline{\Omega})$, has subgradients in the dual space $M(\overline{\Omega})$. Proving $\partial f(0) \neq \emptyset$ requires two arguments. First we establish the existence of a supporting functional. Then we argue that the latter must be continuous since f is lsc.

(b) By the Hahn–Banach theorem, existence of a supporting functional will follow if we show that 0 is an algebraic interior point of $\operatorname{dom} f$. That means for every $\eta \in C(\overline{\Omega})$ we have to find $\rho > 0$ such that $\rho\eta \in \operatorname{dom} f$. Equivalently, we have to show that for every $\eta \in C(\overline{\Omega})$ we can find $\varrho > 0$ such that the equation

$$\operatorname{div} z + \operatorname{div} B^T\lambda - A^T\lambda - \mu - y = \varrho\eta$$

admits a solution (y, z, λ, μ) with $(y, z) \in \operatorname{dom} J_{h^*}$.

As h is not affine, $\operatorname{dom} h^*$ consists of at least two points. By convexity this means that either the projection $\Pi_y(\operatorname{dom}\ h^*)$ of $\operatorname{dom} h^*$ on the first coordinate contains a ball $|y - y_0| \leq \varepsilon$, or that $\Pi_z(\operatorname{dom} h^*)$ contains a segment.

(c) First consider the case where $\Pi_y(\operatorname{dom} h^*)$ has nonempty interior. By convexity there exists an affine function $y \mapsto z(y)$ such that $(y, z(y)) \in \operatorname{dom} h^*$ for all $|y - y_0| \leq \varepsilon$ and some fixed y_0. Let $z(y) = ay + b$, with $a, b \in \mathbb{R}^n$. Setting $y(x) = y_0 + \tilde{y}(x)$, $\mu = -y_0$, and $\lambda = 0$, we have to solve the linear equation

$$a \cdot \nabla\tilde{y} - \tilde{y} = \varrho\eta$$

for $\|\tilde{y}\|_\infty \leq \epsilon$. Assuming without loss that $a_1 \neq 0$, a possible solution is the smooth function

$$\tilde{y}(x) = \varrho\, c(x)\, e^{x_1/a_1}, \quad \text{where} \quad c(x) = \frac{1}{a_1}\int_{\xi_1}^{x_1} \eta(\xi, x_2, \ldots, x_n)\, e^{\xi/a_1}\, d\xi$$

with a suitable $\xi_1 \in \mathbb{R}$. For ϱ sufficiently small we get in fact $\|\tilde{y}\|_\infty \leq \varepsilon$; hence $(\tilde{y}(x), z(\tilde{y}(x))) \in \operatorname{dom} h^*$ for every $x \in \Omega$ and hence $(\tilde{y}, z(\tilde{y})) \in \operatorname{dom} J_{h^*}$ by continuity of $\tilde{y}$. So in the first case the problem is solved.

(d) Now consider the case where $\Pi_y(\operatorname{dom}\ h^*) = \{y_0\}$. Since h is not affine, $\Pi_z(\operatorname{dom} h^*)$ contains at least two points. This means that (eventually with a change of coordinates) $\operatorname{dom} h^*$ contains a convex set of the form

$$\{y_0\} \times \{z_{01}\} \times \cdots \times \{z_{0r}\} \times B_{n-r},$$

with B_{n-r} an open ball with center $(z_{0,r+1}, \ldots, z_{0n})$ in a subspace of dimension $n-r \geq 1$. In the worst case $n - r = 1$, the first $n - 1$ coordinates are fixed, but z_n is free to vary on an interval. Choosing $y \equiv y_0$, $\mu = -y_0$, and $z = z_0 + \tilde{z}$ with $\tilde{z} = (0, \dots, 0, \tilde{z}_n)$, defined by

$$\tilde{z}_n = \varrho \int_{\xi_n}^{x_n} \eta(x_1, \dots, x_{n-1}, \xi)\, d\xi,$$

we get a z having div $z = \varrho\eta$. Also $|z(x) - z_0| \le \varepsilon$ for all $x \in \Omega$ if ϱ is sufficiently small. Then $(y, z) \in$ dom J_{h^*} as required. So in both subcases, 0 is an algebraic interior point of dom f, and a supporting functional at 0 exists. Continuity of the latter follows from the lower semicontinuity of J_{h^*}. This completes the argument started in (a).

(e) Finally, consider the case where h is affine, and hence dom h^* consists of a single point (y_0, z_0). Define the function $f : C(\overline{\Omega}) \to \mathbb{R} \cup \{\infty\}$ as before. Since the value $V(P^*)$ is finite, $0 \in$ dom f, and since the operators A, B have a finite dimensional range, dom f itself is contained in a finite dimensional linear subspace L of $C(\overline{\Omega})$. Linearity of A, B even gives dom$f = L$. Choose a supporting functional at $0 \in L$, and extend it to a continuous linear functional on all of $C(\overline{\Omega})$. ∎

Proposition 4.1 gives existence of a solution of (P^{**}) in $M(\overline{\Omega})$. We argue that under mild additional assumptions, $\bar{u}$ is in fact a function. We will even show a little more, namely, every u feasible for (P^{**}) satisfies $u \in L_\sigma(\Omega)$ for some $\sigma > 1$. Consider $u \in M(\overline{\Omega})$ with

$$\inf_{y,z,\lambda,\mu} L_D(y, z, \lambda, \mu; u) > -\infty,$$

where the infimum is over $y \in C(\bar{\Omega})$, $z \in C^1(\bar{\Omega})^n$, and $\lambda \in \mathbb{R}^N$, $\mu \in \mathbb{R}$ as before. Exploiting the form of L_D leads to three conditions:

$$\inf_{y,z} \left(J_{h^*}(y, z) + \langle u, \operatorname{div} z - y\rangle\right) > -\infty, \tag{4.1}$$

$$\inf_{\lambda} \left(\frac{1}{2\alpha}|\lambda|^2 + \lambda \cdot c + \langle u, \operatorname{div} B^T\lambda - A^T\lambda\rangle\right) > -\infty, \tag{4.2}$$

$$\inf_{\mu} \left(\mu - \langle u, \mu\rangle\right) > -\infty, \quad \text{i.e.,} \int_\Omega du = 1. \tag{4.3}$$

As we shall see, the first condition allows for regularity considerations, while (4.2) and (4.3) will lead back to the original formulation of the constraints in (P).

First consider condition (4.1). We want to show that under suitable assumptions on h every feasible u possesses a Radon–Nikodym derivative lying in every space $L_\sigma(\Omega)$ with $1 < \sigma < \frac{n}{n-1}$. To do this we will need the following estimation for the Newton potential of a function $\varphi \in C_0^\infty(\Omega)$: Let φ be an element of $C_0^\infty(\Omega)$ and consider the corresponding Newton potential

$$v(x) = \int_\Omega \Gamma(x - s)\, \varphi(s)\, ds$$

with

$$\Gamma(x - s) = \begin{cases} \frac{1}{2\pi} \log|x - s|, & n = 2, \\ \frac{1}{n(2-n)\omega_n} |x - s|^{-(n-2)}, & n > 2, \end{cases}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Then we have $v \in C^2(\overline{\Omega})$, $\Delta v = \varphi$, and

$$D_k v(x) = \int_\Omega D_k\Gamma(x - s)\, \varphi(s)\, ds \tag{4.4}$$

(cf. [9, Chapter 4]).

LEMMA 4.2. *Let* $1 < \sigma < \frac{n}{n-1}$. *Then*

$$|D_k v(x)| \le K \, \|\varphi\|_{\sigma'} \quad \text{for all} \quad x \in \Omega \quad \text{and for every} \quad \sigma' > n, \tag{4.5}$$

where the constant K *depends only on* σ' *and* Ω, *and* $1/\sigma + 1/\sigma' = 1$.

Proof. Using (4.4) and Hölder's inequality

$$|D_k v(x)| \le \|D_k \Gamma(x - .)\|_\sigma \, \|\varphi\|_{\sigma'}$$

provided $\|D_k\Gamma(x-.)\|_\sigma$ is finite. But for $n \ge 2$ we have

$$\int_\Omega |D_k\Gamma(x-s)|^\sigma \, ds = C_1 \int_\Omega \left(\frac{|x_k - s_k|}{|x-s|^n} \right)^\sigma ds \le C_2 \int_0^R r^{(n-1)(1-\sigma)} \, dr$$

with $\Omega \subset \{z \in \mathbb{R}^n : |z| \le R\}$ using n-dimensional spherical coordinates. As the last integral is finite for $(n-1)(1-\sigma) > -1$, or what is the same, $\sigma < \frac{n}{n-1}$, the lemma is proved. $\square$

Now we want to use (4.5) and (4.1) to show that the map $\varphi \mapsto \langle u, \varphi \rangle$ is bounded on $C_0^\infty(\Omega)$ with respect to the $\|.\|_{\sigma'}$-norm, hence the Radon–Nikodym derivative of u is an element of $L_\sigma(\Omega)$, $(1/\sigma + 1/\sigma' = 1)$. To do this, we need to impose a richness condition on the domain of h^*:

$$\Pi_z(\text{dom } h^*) \text{ contains a segment.} \tag{$A1$}$$

As before, $\Pi_z : (y, z) \to z$ denotes the projection onto the last n coordinates.

Remark 1. Let us discuss the meaning of $(A1)$. If $\Pi_z(\text{dom } h^*)$ does not contain a segment, dom $h^* \subset \mathbb{R} \times \{z\}$ for some $z \in \mathbb{R}^n$. This implies $h(x, y) = g(x) + y \cdot z$ for a convex function g, that is, h is linear in its second variable. We observe in a first place that z must be in the linear hull of the b_k. Therefore, in cases where we have no constraints on derivatives, $b = 0$ implies $z = 0$, leaving us with a problem without reference to derivatives. In case $b \ne 0$, the problem may be analyzed rather along classical lines as found in [7], although in general the result of Proposition 4.3 below is no longer valid. We consider objectives $h(x, \xi)$ linear in ξ as of minor importance for possible applications and do not pursue this class of objectives any further.

PROPOSITION 4.3. *Under the assumption* $(A1)$ *every* $u \in M(\overline{\Omega})$ *feasible for* (P^{**}) *is absolutely continuous with respect to Lebesgue measure. Its Radon–Nikodym derivative lies in* $L_\sigma(\Omega)$ *whenever* $1 < \sigma < \frac{n}{n-1}$. *Furthermore, for every such* u *there exists a signed Borel vector measure* $m = m(u) \in M(\overline{\Omega})^n$ *satisfying*

$$\langle u, \operatorname{div} z \rangle = -\langle m(u), z \rangle \quad \text{for all} \quad z \in C^1(\overline{\Omega})^n.$$

Remark 2. $m(u)$ is an extension of the distribution vector ∇u on $C(\overline{\Omega})^n$ and shall as well be denoted as ∇u. Notice however that this measure contains singular parts supported on $\partial\Omega$.

Proof. Step 1. Using a reduction argument similar to the one employed in the proof of Proposition 4.1, we may without loss assume that $\Pi_z(\text{dom}h^*)$ has nonempty interior in $\mathbb{R}^n$. The general case consists in repeating the same argument in the affine subspace generated by $\Pi_z(\text{dom}h^*)$, which by $(A1)$ has dimension ≥ 1.

With these arrangements, assumption $(A1)$ guarantees the existence of a ball $|z - z_0| \le \varepsilon$ and an affine function $y = y(z)$ such that $(y(z), z) \in \text{dom } h^*$ for all

$|z - z_0| \le \varepsilon$. Consider $\varphi \in C_0^\infty(\Omega)$ with $\|\varphi\|_{\sigma'} \le \frac{\varepsilon}{K}$ ($\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$) for the constant K from (4.5) and let v be the corresponding Newton potential. Then using (4.5) we have $|D_k v(x)| \le \varepsilon$. Setting $z = z_0 + \nabla v$ and $y = y(z)$ we get from (4.1)

$$J_{h^*}(y(z), z) + \langle u, \varphi - y(z)\rangle > -\infty.$$

Now by construction we have $|J_{h^*}(y(z), z) + \langle u, -y(z)\rangle| \le K_1$ for some $K_1 > 0$, so we get

$$\inf_{\|\varphi\|_{\sigma'} \le \frac{\varepsilon}{K}} \langle u, \varphi\rangle > -\infty.$$

By linearity we conclude that the functional $\varphi \mapsto \langle u, \varphi\rangle$ is bounded on $(C_0^\infty(\Omega), \|.\|_{\sigma'})$ which is a dense subspace of $L_{\sigma'}(\Omega)$. For short $u \in L_\sigma(\Omega)$.

Step 2. For the second statement we have to show that for feasible u the functional $z \mapsto \langle u, \operatorname{div} z\rangle$ is bounded on $(C^1(\overline{\Omega}), \|.\|_\infty)$. This follows from (4.1) and the boundedness of $J_{h^*}(y(z), z)$ and $\langle u, -y(z)\rangle$ on the ball $\|z\|_\infty \le r$. $\square$

For the following suppose condition $(A1)$ is satisfied. In order to simplify our arguments, we continue to consider the case where $\Pi_z(\mathrm{dom} h^*)$ has nonempty interior in $\mathbb{R}^n$. Performing the same steps in the affine subspace L generated by $\mathrm{dom}(h^*)$ will settle the general case.

As a consequence of Propositions 4.1 and 4.3, and on exploiting the structure of L_D, (P^{**}) now reads

$$(P^{**}) \quad \inf_u \Big\{ \sup\left\{\langle u, y\rangle + \langle \nabla u, z\rangle - J_{h^*}(y, z) :\, y \in C(\overline{\Omega}), z \in C^1(\overline{\Omega})^n\right\} \\ + \sup\left\{ -\frac{1}{2\alpha}|\lambda|^2 - \lambda \cdot c + \langle u, A^T\lambda - \operatorname{div} B^T\lambda\rangle : \lambda \in \mathbb{R}^N\right\} : \int_\Omega u(x)\,dx = 1\Big\}.$$

To calculate the inner supremum over y and z we would like to use the following result of Rockafellar's [21] describing the conjugate of a convex integral functional J_{h^*} with respect to the dual pairing $(C(\overline{\Omega}) \times C(\overline{\Omega})^n, M(\overline{\Omega}) \times M(\overline{\Omega})^n)$.

LEMMA 4.4 (see Rockafellar [21]). *Let $\overline{\Omega}$ be a compact subset of $\mathbb{R}^n$ and suppose* $\mathrm{int}(\mathrm{dom}\ h^*) \ne \emptyset$. *Then for $\mu \in M(\overline{\Omega}) \times M(\overline{\Omega})^n$ with Lebesgue decomposition $\mu = \mu_a + \mu_s$ the conjugate of J_{h^*} equals*

$$J^*_{h^*}(\mu) = \int_\Omega h^{**}\left(\frac{d\mu_a}{dx}\right) dx + \int_\Omega \sup_{w \in \mathrm{dom}\, h^*} \left(w \cdot \frac{d\mu_s}{d\vartheta}\right) d\vartheta,$$

where μ_s is absolutely continuous with respect to the nonnegative Borel measure ϑ.

In order to apply Lemma 4.4 to (P^{**}), we first need to replace the supremum over $z \in C^1(\overline{\Omega})$ by a supremum $z \in C(\overline{\Omega})$. That this may be done without changing its value is guaranteed by the following lemma, whose proof will be given in the appendix.

LEMMA 4.5. *Let m be a measure in $M(\overline{\Omega})$, $f \in L_1(\Omega, m)$, $g \in L_1(\Omega, m)^k$ ($k \in \mathbb{N}$), and $\Phi : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$ be a proper convex lsc function. Then for the proper convex lsc functional*

$$F(z) = \int_\Omega [\Phi(z(x))f(x) + z(x) \cdot g(x)]\, dm(x)$$

on $L_1(\Omega, m)^k$ we have

$$\inf_{z \in C^1(\overline{\Omega})^k} F(z) = \inf_{z \in L_1(\Omega, m)^k} F(z).$$

Furthermore, the values of the infima over all function spaces $\mathcal{F}$ with $C^1(\overline{\Omega})^k \subset \mathcal{F} \subset L_1(\Omega, m)^k$ agree.

Applying Lemma 4.5 to (P^{**}) with $k = n+1$, letting $z(x)$ stand for the pair $(y(x), z(x))$, $g(x) = \nabla u(x)$, and letting $\Phi(z(x))f(x)$ represent the term $h^*(y(x), z(x)) + u(x)y(x)$, we are now allowed to calculate

$$\sup_{\substack{y \in C(\overline{\Omega}) \\ z \in C(\overline{\Omega})^n}} \left(\langle u, y \rangle + \langle \nabla u, z \rangle - J_{h^*}(y, z) \right) \tag{4.6}$$

in (P^{**}), and Lemma 4.4 then shows that (4.6) equals

$$J_{h^{**}}(u, (\nabla u)_a) + \int_\Omega \sup_{z \in \Pi_z(\operatorname{dom} h^*)} \left(z \cdot \frac{d(\nabla u)_s}{d\vartheta}(x) \right) d\vartheta(x), \tag{4.7}$$

where $\nabla u = (\nabla u)_a + (\nabla u)_s$ denotes Lebesgue decomposition and $d(\nabla u)_s \ll d\vartheta$. A possible choice for ϑ is, for instance, the total variation of $d(\nabla u)_s$. For every feasible u we get in particular

$$\int_\Omega \sup_{z \in \Pi_z(\operatorname{dom} h^*)} \left(z \cdot \frac{d(\nabla u)_s}{d\vartheta}(x) \right) d\vartheta(x) = \int_\Omega \sigma_{\Pi_z(\operatorname{dom} h^*)} \left(\frac{d(\nabla u)_s}{d\vartheta}(x) \right) d\vartheta(x) < \infty. \tag{4.8}$$

Here $\sigma_{\Pi_z(\operatorname{dom} h^*)}(y)$ denotes the support function of the convex set $\Pi_z(\operatorname{dom} h^*)$ (cf. [21]).

Example 2. For the Csiszár information measures (1.4) we have $\Pi_z(\operatorname{dom} h^*) = \operatorname{dom} \phi^*$. From [21, Theorem 13.3] we deduce $\sigma_{\operatorname{dom} \phi^*} = \phi^{0+}$, the recession function of ϕ:

$$\phi^{0+}(y) = \lim_{\lambda \to \infty} \frac{1}{\lambda} (\phi(x + \lambda y) - \phi(x)) \quad \text{for an arbitrary} \quad x \in \operatorname{dom} \phi.$$

For the particular case $\phi(t) = |t|^p$, $p > 1$, we have

$$\phi^{0+}(y) = \begin{cases} \infty & \text{if} \ \ y \neq 0, \\ 0 & \text{if} \ \ y = 0, \end{cases}$$

while the case $p = 1$, $\phi(t) = |t|$ gives $\phi^{0+}(y) = |y|$. So for $p > 1$ the singular part of ∇u in (4.7) must vanish, since $\phi^{0+}((d(\nabla u)_s/d\vartheta)(x)) < \infty$ only for $(d(\nabla u)_s/d\vartheta)(x) = 0$ a.e. On the other hand, in case $p = 1$ we cannot argue that $(\nabla u)_s = 0$, so we only get $u \in BV(\Omega)$.

In general we need the assumption

$$\Pi_z(\operatorname{dom} h^*) \text{ is an affine subspace of dimension } \geq 1 \tag{$A2$}$$

to get $u \in W^{1,1}(\Omega)$. Notice that $(A2)$ readily implies $(A1)$. To understand the meaning of $(A2)$, consider the case where $\Pi_z(\operatorname{dom} h^*) = \mathbb{R}^n$. Then $h(x, \xi)$ is coercive in its ξ-variable. More precisely, $\Pi_z(\operatorname{dom} h^*) = \mathbb{R}^n$ implies that for every fixed x, $h(x, \xi)$ grows stronger than linearly as $|\xi| \to \infty$.

PROPOSITION 4.6. *If assumption $(A2)$ is satisfied, every u which is feasible for (P^{**}) lies in $W^{1,1}(\Omega)$, and (P^{**}) has the form*

$$\inf_{u \in W^{1,1}(\Omega)} \left\{ I_h[u] + \frac{\alpha}{2} |Au + B\nabla u - c|^2 \ : \ \int_\Omega u(x)\, dx = 1 \right\}. \tag{P^{**}}$$

Remark 3. This means that (P^{**}) coincides with $(P)_{\text{pen}}$, formulated in the Sobolev space $W^{1,1}(\Omega)$. Notice again here that this does not exclude situations where the natural space for the objective $I_h[u]$ is smaller, e.g., $\text{dom} I_h \subset W^{1,2}(\Omega)$. In this case, $u \notin W^{1,2}(\Omega)$ will have $I_h[u] = \infty$ and the solution will automatically be an element of $W^{1,2}(\Omega)$.

Proof. As before, we present the argument in the case $\Pi_z(\text{dom} h^*) = \mathbb{R}^n$, i.e., where the affine subspace L generated by dom h^* has dimension n. The general case is settled by repeating the argument in L.

Under these circumstances, condition $(A2)$ in tandem with $h = h^{**}$ allows reducing (4.7) to $J_h(u, \nabla u) = I_h[u]$. Indeed, with $z \in \Pi_z(\text{dom}\, h^*)$ arbitrary, the supremum under the integral sign in (4.7) is $+\infty$, unless $(\nabla u)_s = 0$. Hence $(\nabla u)_a = \nabla u$, and the claim follows. Further, we can write (4.2) in the form

$$\sup_{\lambda \in \mathbb{R}^N} \left(-\frac{1}{2\alpha}|\lambda|^2 - \lambda \cdot c + \langle u, A^T\lambda - \text{div}\, B^T\lambda \rangle \right)$$
$$= \sup_{\lambda \in \mathbb{R}^N} \left(\lambda \cdot (Au + B\nabla u - c) - \frac{\alpha}{2}|\lambda|^2 \right) = \frac{1}{2\alpha}|Au + B\nabla u - c|^2,$$

so (P^{**}) is $(P)_{\text{pen}}$ formulated in the space $W^{1,1}(\Omega)$. □

Propositions 4.3 and 4.6 now yield the main result for $(P)_{\text{pen}}$.

THEOREM 4.7. *Under the hypothesis* $(A2)$, *the penalization model* $(P)_{\text{pen}}$ *admits a solution* $\overline{u} \in W^{1,1}(\Omega)$.

5. Existence of solutions for $(P)_{\text{tol}}$. Similar to $(P)_{\text{pen}}$, the tolerance model $(P)_{\text{tol}}$ can be written in the form

$$\inf_{u,v,e} \sup_{w,\lambda,\mu,\nu \geq 0} \tilde{L}(u, v, e; w, \lambda, \mu, \nu)$$

with

$$\tilde{L}(u, v, e; w, \lambda, \mu, \nu) = J_h(u, v) + \langle w, \nabla u - v \rangle + \lambda \cdot (Au + Bv - c - e)$$
$$+\mu \left(\int_\Omega u(x)\, dx - 1 \right) + \nu(|e|^2 - \varepsilon^2).$$

Here we get the analogous results by similar reasoning so we will only cite the main theorem.

THEOREM 5.1. *If* $(A2)$ *is satisfied, the tolerance model* $(P)_{\text{tol}}$ *admits a solution* $\overline{u} \in W^{1,1}(\Omega)$.

6. Regularity. In this section we show that the regularity of the solutions $\bar{u}$ of $(P)_{\text{pen}}$ and $(P)_{\text{tol}}$ may be improved to give $\bar{u} \in W^{1,p}(\Omega)$ for some $p > 1$ if condition $(A2)$ is strengthened. For $1 \leq \varrho \leq r$ consider the condition

$$(A3) \qquad \begin{array}{l} \text{there exists a measurable function } y \mapsto y(z),\ \mathbb{R}^n \to \mathbb{R}, \text{ such that} \\ \text{(i)} \quad |y(z)| \leq K(1 + |z|^\varrho) \text{ for every } z \in \mathbb{R}^n, \text{ and} \\ \text{(ii)} \quad J_{h^*}(y(z), z) \text{ is bounded on a ball } \{z \in C^1(\overline{\Omega}) : \|z\|_r \leq C\}. \end{array}$$

Clearly $(A3)$ implies $(A2)$ and may be understood as a coercivity condition on the integrand h. Notice that $(A3)$ is, for instance, satisfied if h^* satisfies the growth condition

$$\text{(ii}'\text{)} \quad h^*(y, z) \leq K(1 + |y|^{r/\varrho} + |z|^r),$$

which translates into a coercivity condition for h. In this case, $y(z) = |z|^\rho$ satisfies (i).

COROLLARY 6.1. *Suppose (A3) (with $1 \le \varrho \le r$) is satisfied. Then every u feasible for $(P)_{\mathrm{pen}}$ (resp., $(P)_{\mathrm{tol}}$) lies in $W^{1,p(r,\varrho)}(\Omega)$ with*

$$p(r,\varrho)\begin{cases} = \infty, & \varrho = r = 1, \\ = \frac{r}{r-1}, & \varrho = 1 < r, \\ = \frac{r}{r-1}, & \varrho > 1 \text{ and } r > n(\varrho - 1), \\ < \frac{n(\varrho-1)}{n(\varrho-1)-1}, & \varrho > 1 \quad \text{and} \quad r \le n(\varrho - 1). \end{cases}$$

In particular, this is true for the solution $\overline{u}$ of $(P)_{\mathrm{pen}}$ or $(P)_{\mathrm{tol}}$.

For the proof we need the following.

LEMMA 6.2. *Given $u \in W^{1,1}(\Omega)$ with $\nabla u \in L_p(\Omega)^n$ for some $p > 1$ we have $u \in W^{1,p}(\Omega)$.*

Proof. We have to show $u \in L_p(\Omega)$. Consider the sequence

$$u_n(x) = \begin{cases} u(x), & |u(x)| \le n, \\ n, & u(x) > n, \\ -n, & u(x) < -n. \end{cases}$$

Then [28, Corollary 2.1.8] gives

$$\nabla u_n(x) = \begin{cases} \nabla u(x), & |u(x)| < n, \\ 0, & |u(x)| \ge n, \end{cases}$$

hence $u_n \in W^{1,p}(\Omega)$ for all n. We want to show that $\|u_n\|_p$ is bounded so the Fatou lemma will give the result. Following [1, Theorem 4.20] for each $\varepsilon > 0$ there exists a set $\Omega_\varepsilon \subset\subset \Omega$ such that for every $v \in W^{1,p}(\Omega)$,

$$\|v\|_p \le K\varepsilon \|\nabla v\|_p + K\|v\|_{p,\Omega_\varepsilon}$$

with $K = K(p, \Omega)$. (Here $\|v\|^p_{p,\Omega_\varepsilon} = \int_{\Omega_\varepsilon} |v(x)|^p\,dx$.) Now since $u \in L^{\mathrm{loc}}_p(\Omega)$ (cf. [11, Theorem 4.5.13]) we have $\|u\|_{p,\Omega_\varepsilon} < \infty$, and from the definition of u_n we get

$$\|u_n\|_p \le K\varepsilon\|\nabla u_n\|_p + K\|u_n\|_{p,\Omega_\varepsilon} \le K\varepsilon\|\nabla u\|_p + K\|u\|_{p,\Omega_\varepsilon} = C < \infty$$

for every n. Now, using $u_n(x) \to u(x)$ a.e., Fatou's lemma provides

$$\|u\|_p \le \liminf_{n\to\infty} \|u_n\|_p \le C,$$

hence $u \in L_p(\Omega)$. $\square$

We proceed to complete the proof of the corollary.

Proof. We give the argument for $(P)_{\mathrm{pen}}$, the tolerance case being similar. Suppose $u \in W^{1,1}(\Omega)$ is feasible for $(P)_{\mathrm{pen}}$. By Proposition 4.6, it is then feasible for (P^{**}) as well, so we have

$$\inf_{z \in C^1(\overline{\Omega})^n} \left(J_{h^*}(y(z), z) + \langle u, y(z)\rangle + \langle \nabla u, z\rangle\right) > -\infty. \tag{6.1}$$

We want to construct a decreasing (possibly breaking off) sequence of exponents $r_k \ge r$ such that the term $J_{h^*}(y(z), z) + \langle u, y(z)\rangle$ is bounded on a ball $\|z\|_{r_k} \le C$, giving $\nabla u \in L_{p_k}(\Omega)$ with $p_k = \frac{r_k}{r_k - 1}$ by (6.1). Lemma 6.2 will imply $u \in W^{1,p_k}(\Omega)$.

The procedure is the following. Suppose we have already constructed $r_k > r, p_k = r_k/(r_k-1) > 1$, with $u \in W^{1,p_k}(\Omega)$ by the argument above. Then by the Sobolev embedding theorem (cf. [1, p. 97]) we have $u \in L_{s_k}(\Omega)$ with

$$s_k \begin{cases} \text{arbitrary if } \ p_k \geq n, \\ = \frac{np_k}{n-p_k} \ \text{ if } \ p_k < n. \end{cases}$$

Using Hölder's inequality and condition (i) of $(A3)$ we have

$$|\langle u, y(z)\rangle| \leq \|u\|_{s_k} \|y(z)\|_{s_k'} \leq \tilde{K}\, \|u\|_{s_k} \|z\|_{\varrho s_k'}$$

with $\frac{1}{s_k} + \frac{1}{s_k'} = 1$. On the other hand, condition (ii) of $(A3)$ implies that $J_{h^*}(y(z), z)$ is bounded on $\|z\|_r \leq C$. Hence we choose $r_{k+1} = \max(r, \varrho s_k')$. By construction the sequence (r_k) is strictly decreasing unless $r_k = r$, the lowest exponent we can possibly achieve. As soon as $r_k = r$ for some index k, the process stops giving $u \in W^{1,\frac{r}{r-1}}(\Omega)$. (Notice that if $p_k \geq n$ for some k, we can always choose s_k large enough to guarantee $\varrho s_k' \leq r$, viz. $r_{k+1} = r$.)

Now we want to compute the r_k explicitly: From Theorem 4.7 we know $u \in W^{1,1}(\Omega)$, i.e., $p_0 = 1$, giving $s_1 = \frac{n}{n-1}$ and $r_1 = \max(r, \varrho n)$. So we can actually stop after the first step if $r \geq \varrho n$. Otherwise we get $p_1 = \frac{\varrho n}{\varrho n - 1}$,

$$\begin{cases} s_1 = \frac{n\varrho}{n\varrho - \varrho - 1} \ \text{ and } \ r_2 = \max(r, \frac{n\varrho^2}{1+\varrho}) & \text{for } \ \frac{n\varrho}{n\varrho-1} < n, \\ r_2 = r & \text{if } \ \frac{n\varrho}{n\varrho - 1} \geq n. \end{cases}$$

Proceeding like this we get

$$r_k = \begin{cases} \frac{n}{k} \to 0 & \text{if } \ \varrho = 1, \\ n\frac{\varrho - 1}{1-\varrho^{-(k+1)}} \searrow n(\varrho - 1) & \text{if } \ \varrho > 1, \end{cases}$$

so the process will break off, giving $u \in W^{1,\frac{r}{r-1}}(\Omega)$, unless $\varrho > 1$ and $r \leq n(\varrho - 1)$. In the latter case we still get $u \in W^{1,p}(\Omega)$ for every $p < \frac{n(\varrho-1)}{n(\varrho-1)-1}$. $\square$

Example 3. For the Csiszár information measures with $\phi(t) = |t|^p$ discussed in the preparatory section 1, we may choose $y(z) = K|z|^{p'}$ $(\frac{1}{p} + \frac{1}{p'} = 1)$, so $r := \varrho := p'$ will do, and the corollary gives $u \in W^{1,\tilde{p}}(\Omega)$ with

$$\tilde{p} \begin{cases} = p, & p > n, \\ < \frac{n}{n-p+1} = 1 + \frac{p-1}{n-p+1}, & p \leq n. \end{cases}$$

In particular, for p close to 1 we have $\tilde{p} = 1 + \epsilon(p)$ and $u \in W^{1,1+\varepsilon(p)}(\Omega)$ with $\varepsilon(p) := \frac{p-1}{n-p+1} \downarrow 0$ for $p \downarrow 1$.

Appendix. We still have to prove Lemma 4.5.

LEMMA 4.5. *Let m be a measure in $M(\overline{\Omega})$, $f \in L_1(\Omega, m)$, $g \in L_1(\Omega, m)^k$ $(k \in \mathbb{N})$, and $\Phi : \mathbb{R}^k \to \mathbb{R} \cup \{\infty\}$ be a proper convex lsc function. Then for the proper convex lsc functional*

$$F(z) = \int_\Omega \left[\Phi(z(x)) f(x) + z(x) \cdot g(x)\,\right] dm(x)$$

on $L_1(\Omega, m)^k$ we have

$$\inf_{z \in C^1(\overline{\Omega})^k} F(z) = \inf_{z \in L_1(\Omega,m)^k} F(z).$$

Furthermore, the values of the infima over all function spaces $\mathcal{F}$ with $C^1(\overline{\Omega})^k \subset \mathcal{F} \subset L_1(\Omega, m)^k$ agree.

Proof. We give the argument in the case where dom Φ has nonempty interior in $\mathbb{R}^k$, the general case being reducible to the former.

It is sufficient to show that for any $z \in L_1(\Omega,\mu)^m$ with $F(z) < \infty$ and any $\varepsilon_0 > 0$ we can find a $y \in C^1(\overline{\Omega})^m$ such that $F(y) \leq F(z) + \varepsilon_0$. The construction of such a y will be divided into three steps. First we will prove the existence of a function $\tilde{z} \in L_1(\Omega,\mu)^m$ with values only in int(dom Φ) having $F(\tilde{z}) \leq F(z) + \frac{\varepsilon_0}{3}$. Then we will modify $\tilde{z}$ to get a function $\tilde{z}_{n_0}$ which maps Ω into a compact subset of int(dom Φ), again having $F(\tilde{z}_{n_0}) \leq F(\tilde{z}) + \frac{\varepsilon_0}{3}$. For the last step we will use the Lipschitz continuity of Φ on compact subsets of int(dom Φ) to find a suitable measure ν such that the approximation of $\tilde{z}_{n_0}$ with respect to ν by $C^1(\overline{\Omega})^m$-functions will also approach $F(\tilde{z}_{n_0})$. This will finally prove the existence of a $y \in C^1(\overline{\Omega})^m$ with $F(y) \leq F(\tilde{z}_{n_0}) + \frac{\varepsilon_0}{3}$, giving $F(y) \leq F(z) + \varepsilon_0$ altogether.

Step 1. Consider $z \in L_1(\Omega,\mu)^m$ with $F(z) < \infty$. Then we have $z(x) \in \operatorname{dom}\Phi$ a.e. for every representative of z. In particular, we can choose a measurable representative also denoted by z having $z(x) \in \operatorname{dom}\Phi$ for all $x \in \Omega$.

For fixed $\varepsilon > 0, \delta > 0$ we define the set-valued mapping

$$\Gamma(x) = \{\zeta \in \text{int}(\text{dom }\ \Phi) : |\zeta - z(x)| \leq \varepsilon,\ \Phi(\zeta) \leq \Phi(z(x)) + \delta\} \ \text{ for all }\ x \in \Omega.$$

We want to show that Γ admits a measurable selector using the Kuratovski and Ryll-Nardczevski measurable selection theorem [14]: Suppose Γ is a measurable set-valued mapping with nonempty closed images. Then there exists a measurable $\tilde{z} : \Omega \to \mathbb{R}^m$ having $\tilde{z}(x) \in \Gamma(x)$ for every $x \in \Omega$.

As will be seen, for sufficiently small ε, δ, this selector satisfies $F(\tilde{z}) \leq F(z) + \frac{\varepsilon_0}{3}$.

We have to verify three properties of Γ:

$\Gamma(x)$ *is closed for every* $x \in \Omega$: Fix $x \in \Omega$ and suppose (ζ_n) is a sequence in $\Gamma(x)$ converging to some $\zeta \in$ int(dom Φ). Then we have $|\zeta - z(x)| = \lim_{n\to\infty} |\zeta_n - z(x)| \leq \varepsilon$ and $\Phi(\zeta) \leq \liminf_{n\to\infty} \Phi(\zeta_n) \leq \Phi(z(x)) + \delta$, so $\zeta \in \Gamma(x)$ and $\Gamma(x)$ is closed in int(dom Φ).

$\Gamma(x)$ *is nonempty for all* $x \in \Omega$: Since Φ is proper convex and lsc, epi Φ is a closed convex subset of $\mathbb{R}^m \times \mathbb{R}$ with

$$\operatorname{epi}\Phi = \overline{\operatorname{int}(\operatorname{epi}\Phi)}$$

(cf. [21, p. 46]). Hence any point $(z(x), \Phi(z(x))) \in \operatorname{epi}\Phi$ can be approximated by a sequence $(\zeta_n, \Phi(\zeta_n) + \delta_n) \in$ int(epi Φ); that means $\zeta_n \in$ int(dom Φ) and $\delta_n > 0$. But then we must have $\zeta_n \in \Gamma(x)$ for n sufficiently large, so $\Gamma(x)$ is nonempty.

Γ *is measurable:* We have to show that for each measurable $M \subset \mathbb{R}^m$ the preimage $\Gamma^{-1}(M)$ is a measurable subset of Ω. Here, without loss of generality, we can assume $M \subset$ int(dom Φ). We get

$$\begin{aligned}
\Gamma^{-1}(M) &= \big\{x \in \Omega : \exists\, \zeta \in M : z(x) \in B(\zeta,\varepsilon), \Phi(z(x)) \geq \Phi(\zeta) - \delta\big\} \\
&= \big\{x \in \Omega : \exists\, y \in B(0,\varepsilon),\, \exists\, \zeta \in M : z(x) = y + \zeta,\ \Phi(z(x)) \geq \Phi(\zeta) - \delta\big\} \\
&= \big\{x \in \Omega : \exists\, y \in B(0,\varepsilon) \colon (z(x), \Phi(z(x))) \in \operatorname{epi}(\Phi - \delta) \cap (M \times \mathbb{R}) + \{(y,0)\}\big\} \\
&= \big\{x \in \Omega : (z(x), \Phi(z(x))) \in \operatorname{epi}(\Phi - \delta) \cap (M \times \mathbb{R}) + B(0,\varepsilon) \times \{0\}\big\} \\
&= (z, \Phi(z))^{-1} \left(\operatorname{epi}(\Phi - \delta) \cap (M \times \mathbb{R}) + B(0,\varepsilon) \times \{0\}\right).
\end{aligned}$$

Since epi $(\Phi - \delta)$ is closed and therefore measurable, $\Gamma^{-1}(M)$ is now the preimage of a measurable set under the measurable map $x \mapsto (z(x), \Phi(z(x)))$; hence $\Gamma^{-1}(M)$ is measurable.

Now we can apply the selection theorem of Kuratovski and Ryll-Nardczevski to get a measurable function $\tilde{z}$ with $\tilde{z}(x) \in \Gamma(x)$ for all $x \in \Omega$. By the definition of Γ we have $\tilde{z} \in L_1(\Omega, \mu)^m$ and

$$F(\tilde{z}) \leq F(z) + \delta \int_\Omega |f(x)|\, d\mu(x) + \varepsilon \int_\Omega |g(x)|\, d\mu(x),$$

so for sufficiently small δ and ε, $F(\tilde{z}) \leq F(z) + \frac{\varepsilon_0}{3}$, and the first step is proven.

Step 2. Without loss of generality, assume $0 \in \operatorname{int}(\operatorname{dom} \Phi)$ and $\Phi(0) = 0$. Now choose an increasing sequence (D_n) of compact subsets of $\mathbb{R}^m$ having $0 \in D_1$ and

$$D_n \uparrow \operatorname{int}(\operatorname{dom}\ \Phi),\ D_n \subseteq \{\zeta \in \mathbb{R}^m : |\Phi(\zeta)| \leq n\},\ D_n \subset \operatorname{int}(D_{n+1}).$$

Defining $\Omega_n = \tilde{z}^{-1}(D_n) = \{x \in \Omega : \tilde{z}(x) \in D_n\}$ and letting $\tilde{z}_n = \chi_{\Omega_n} \cdot \tilde{z}$, we have $\tilde{z}_n(x) \to \tilde{z}(x)$ and $\Phi(\tilde{z}_n(x)) \to \Phi(\tilde{z}(x))$ pointwise, since $\Omega_n \uparrow \Omega$. Now

$$|F(\tilde{z}_n)| \leq \int_\Omega |\Phi(\tilde{z}(x)) f(x) + \tilde{z}(x) \cdot g(x)|\, d\mu(x) \quad \text{for all} \quad n \in \mathbb{N}$$

and dominated convergence implies $F(\tilde{z}_n) \to F(\tilde{z})$, so we can choose some $n_0 \in \mathbb{N}$ with $F(\tilde{z}_{n_0}) \leq F(\tilde{z}) + \frac{\varepsilon_0}{3}$.

Step 3. Now we have $\tilde{z}_{n_0}(x) \in D_{n_0}$ for all $x \in \Omega$, so if we want to approach $\tilde{z}_{n_0}$ by smooth functions we can restrict ourselves to functions with values in D_{n_0+1} since D_{n_0} is a compact subset of $\operatorname{int}(D_{n_0+1})$ having a positive distance from its boundary. But for each $y \in C^1(\overline{\Omega})^m$ with values in D_{n_0+1} we get

$$|F(\tilde{z}_{n_0}) - F(y)| \leq \int_\Omega |\tilde{z}_{n_0}(x) - y(x)|\ (L_{n_0+1}|f(x)| + |g(x)|)\ d\mu(x),$$

where L_{n_0+1} denotes the Lipschitz constant of Φ on D_{n_0+1}. So we have to approximate $\tilde{z}_{n_0}$ with respect to the measure $d\nu = (L_{n_0+1}|f| + |g|)\, d\mu$. Choosing $y \in C^1(\overline{\Omega})^m$ with $\|\tilde{z}_{n_0} - y\|_{L_1(\Omega,\nu)^m} \leq \frac{\varepsilon_0}{3}$ (notice $\tilde{z}_{n_0} \in L_1(\Omega, \nu)^m$, hence such a y exists), we finally get

$$F(y) \leq F(\tilde{z}_{n_0}) + \frac{\varepsilon_0}{3} \leq F(\tilde{z}) + 2\frac{\varepsilon_0}{3} \leq F(z) + \varepsilon_0,$$

and the proof is complete. $\square$

REFERENCES

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, NY, 1978.

[2] G. Aubert and L. Vese, *A variational method in image recovery*, SIAM J. Numer. Anal., 34 (1997), pp. 1948–1979.

[3] J.M. Borwein and A.S. Lewis, *Partially-finite programming in L_1 and the existence of maximum entropy estimates*, SIAM J. Optim., 3 (1993), pp. 248–267.

[4] J.M. Borwein, A.S. Lewis, and D. Noll, *Maximum entropy reconstruction using derivative information* I: *Fisher information and convex duality*, Math. Oper. Res., 21 (1996), pp. 442–468.

[5] T. Chan and L. Vese, *Variational Image Restoration and Segmentation Models and Approximations*, Technical Report CAM 97-47, University of California, Los Angeles, 1997.

[6] D. Dobson and F. Santosa, *Recovery of blocky images from noisy and blurred data*, SIAM J. Appl. Math., 56 (1996), pp. 1181–1198.

[7] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland Publishing Company, Amsterdam, 1976.
[8] S. Geman and D. Geman, *Stochastic relaxation, Gibbs distribution and the Bayesian restoration of images*, IEEE Trans. Pattern Anal. Machine Intell., 6 (1984), pp. 721–741.
[9] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1977.
[10] U. Hermann, *Rekonstruktionsprobleme bei unvollständiger Information: Funktionale vom erweiterten Entropietyp*, Doctoral thesis, Mathematisches Institut A, Universität Stuttgart, Stuttgart, 1997, Shaker-Verlag, Aachen.
[11] L. Hörmander, *The Analysis of Linear Partial Differential Operators* I, Springer-Verlag, New York, 1990.
[12] K. Ito and K. Kunisch, *An active set strategy based on the augmented Lagrangian formulation for image restoration*, RAIRO Modél. Math. Anal. Numér, to appear.
[13] K.E. Casas, K. Kunisch, and C. Pola, *Regularization by Functions of Bounded Variation and Applications to Image Enhancement*, Technical Report 1996.5, Dpto. Mat. Est. y Comp., Universidad de Cantabria, Spain, 1996.
[14] K. Kuratovski and C. Ryll-Nardczevski, *A general theorem on selectors*, Bull. Acad. Pol. Sci. Sér. Sci. Math. Astron. Phys., 13 (1965), pp. 397–403.
[15] Y. Li and F. Santosa, *An affine scaling algorithm for minimizing total variation in image enhancement*, Technical Report CTC94TR201, Cornell Theory Center, Cornell University, Ithaca, NY, 1994.
[16] A.K. Louis, *Inverse und schlecht gestellte Probleme*, Teubner-Verlag, Leipzig, 1989.
[17] J.-P. Morel and S. Solimini, *Variational Methods in Image Segmentation*, Progr. Nonlinear Differential Equations Appl. 14, Birkhäuser-Verlag, Basel, Switzerland, 1995.
[18] D. Noll, *Restoration of degraded images with maximum entropy*, J. Global Optim., 10 (1997), pp. 91–103.
[19] D. Noll, *Reconstruction with noisy data—an approach via eigenvalue optimization*, SIAM J. Optim., 8 (1998), pp. 82–104.
[20] D. Noll, *Variational methods in image restoration*, in Recent Advances in Optimization, Lecture Notes in Econom. and Math. Systems 452, P. Gritzmann, R. Horst, E. Sachs, and R. Tichatschke, eds., Springer-Verlag, Berlin, 1997, pp. 229–245.
[21] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.
[22] R.T. Rockafellar, *Integrals which are convex functionals*, I and II, Pacific J. Math., 24 (1968), pp. 525–539, and 39 (1971), pp. 439–469.
[23] L. Rudin and S. Osher, *Total variation based image restoration with free local constraints*, Proc. IEEE ICIP, I (1994), pp. 31–35.
[24] S. Osher and L. Rudin, *Feature-oriented image enhancement using shock filters*, SIAM J. Numer. Anal., 27 (1990), pp. 919–940.
[25] L. Rudin, S. Osher, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Phys. D, 60 (1992), pp. 259–268.
[26] L. Vese, *A study in the BV space of a denoising-deblurring variational problem*, to appear.
[27] L. Vese and T. Chan, *Reduced non-convex functional approximations for image restoration and segmentation*, Technical Report CAM 97-56, University of California, Los Angeles, 1997.
[28] W.P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York, 1989.