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A Unified Approach to Duality in Convex Programming

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Bifunctional Duality, Lagrange Duality, and Fenchel Duality in convex programming are presented under a common point of view. Stability criteria, weaker and more natural than the usual Slater type constraint qualification hypotheses, are obtained for any of these types of duality. © 1991 Academic Press, Inc.

1. INTRODUCTION

The purpose of this note is to present Bifunctional Duality, Lagrange Duality, and Fenchel Duality in convex programming from a common point of view. As a result, we obtain stability criteria for each of the duality concepts weaker and more natural than the usual Slater type constraint qualification hypotheses. We do this by showing that the three types of duality are actually equivalent, and then use the well-known description of Fenchel duality in terms of the Sandwich Theorem. The natural requirements needed to prove versions of the Sandwich Theorem (cf. [1, 4, 5, 6]) then lead to the mentioned stability criteria. For instance, using this reduction, we obtain Rockafellar's stability condition in Lagrange duality (cf. [7, p. 192; 8]).

The interrelation between different types of duality has been investigated by various authors. We just mention Refs. [2, 7, 8, 12], where the case of Slater type conditions is treated. Our access provides stability conditions of a fairly general type, covering all cases of practical relevance.

2. YOUNG–FENCHEL TRANSFORM

Let E, F be normed spaces, and let F be ordered by a closed convex and normal cone F_+ (cf. [9, p. 215]). Throughout we assume that F is order-complete, which means that subsets of F which are order-bounded below admit infima (see [9, p. 209]).

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Let $\phi: E \to F \cup \{\infty\}$ be a convex operator. The *Young–Fenchel transform* ϕ^* of ϕ is defined by

$$\phi^*(f) = \sup_{x \in E} (f(x) - \phi(x)),$$

$f \in \mathscr{L}(E, F)$, where $\mathscr{L}(E, F)$ denotes the space of continuous linear operators from E to F. In the case where the domain

$$D(\phi) = \{x \in E: \phi(x) < \infty\}$$

of ϕ is nonempty, $\phi^*: \mathscr{L}(E, F) \to F \cup \{\infty\}$ is a convex operator. $D(\phi^*)$ may be empty, however; i.e., we may have $\phi^* \equiv \infty$. If ϕ is lower semi-continuous with values in $\mathbb{R} \cup \{\infty\}$, the Brøndsted–Rockafellar Theorem implies that $\partial\phi(x) \neq \emptyset$on a dense set of points $x \in D(\phi)$, from which we readily deduce that $D(\phi^*) \neq \emptyset$. In the vector-valued case, no analogue of the Brøndsted–Rockafellar Theorem is available, but a generalization of Hörmander's theorem, proved in [4, Section 2], yields $D(\phi^*) \neq \emptyset$, at least in the case of a lower semi-continuous ϕ having values in $F \cup \{\infty\}$ for an order-complete F having an order-unit (cf. [9, p. 205]). Clearly the case where $\operatorname{int} D(\phi) \neq \emptyset$ causes no difficulties, for then the Hahn–Banach Theorem proves $\partial\phi(x) \neq \emptyset$ on $\operatorname{int} D(\phi)$, from which we obtain $D(\phi^*) \neq \emptyset$ (see [11]). The general vector-values case, however, is more complicated, as the following example indicates.

EXAMPLE. Let $E = l^1(\mathbb{N})$, $F = l^2(\mathbb{N})$, the latter ordered in the natural way. Let $D(\phi)$ be the cone of $x \in l^2(\mathbb{N})$ having $x_n \geqslant 0$, and define ϕ by

$$\phi(x) = (-\sqrt{x_n})_{n=1}^{\infty}.$$

Then ϕ is even Lipschitz continuous, convex, and defined on a closed convex and generating cone $D(\phi)$ in $l^1(\mathbb{N})$. But $\phi^* \equiv \infty$ here, as may be seen by [4, Example 1].

If the domain $D(\phi)$ of ϕ satisfies some mild completeness condition, the statement $D(\phi^*) \neq \emptyset$ may be obtained even under weaker requirements on the operator ϕ provided that F has an order-unit. In this case mere measurability assumptions on ϕ are sufficient.

PROPOSITION 1. *Let E be a Banach space, F a normally ordered normed space. Suppose F is order-complete and has an order-unit. Let $\phi: E \to F \cup \{\infty\}$ be a convex operator. Suppose* (i) *$D(\phi)$ is CS-closed* (*cf.* [3]), *and* (ii) *for every $f \in F'$, $f \geqslant 0$, the convex functional $f \circ \phi$ is majorized by a Borel measurable mapping $\psi_f: D(\phi) \to \mathbb{R}$. Then $D(\phi^*) \neq \emptyset$.*

The proof is essentially the same as in [4, Theorem 7], where we proved that under the assumptions (i), (ii) above the operator ϕ admits a continuous affine support mapping $h: E \to F$, $h \leqslant \phi$. Clearly, $h = f + v$ for certain $f \in \mathscr{L}(E, F)$, $v \in F$, whence $f \in D(\phi^*)$.

3. Bifunctional Duality

Let E, F be given as in Section 2, and let $\phi: E \to F \cup \{\infty\}$ be a convex operator with nonempty domain. A convex *bifunction* $\Phi: E \times Y \to F \cup \{\infty\}$ is called a *perturbation* of ϕ if $\Phi(\cdot, 0) = \phi$. The normed space Y is referred to as the perturbation space. We consider the convex optimization problem

$$(\mathrm{P})_B \qquad \text{minimize} \quad \phi(x) = \Phi(x, 0), \qquad x \in E.$$

The value v_B of $(\mathrm{P})_B$ is $v_B = \inf\{\phi(x): x \in E\} \in F \cup \{-\infty\}$.

We consider the associated dual optimization problem

$$(\mathrm{P}^*)_B \qquad \text{maximize} \quad -\Phi^*(0, f), \qquad f \in \mathscr{L}(Y, F),$$

where Φ^* denotes the Young–Fenchel transform of Φ, and where $(0, f)$ stands for the linear operator $(x, y) \to f(y)$ on $E \times Y$. Let $w_B = \sup\{-\Phi^*(0, f): f \in \mathscr{L}(Y, F)\}$ be the value of the dual problem $(\mathrm{P}^*)_B$, then we have the well-known

Proposition 2. $-\infty \leqslant w_B \leqslant v_B$.

This is immediate from the estimate

$$-\Phi^*(0, f) \leqslant \inf_{x, y} (\Phi(x, y) - f(y)) \leqslant \inf_x \Phi(x, 0) = v_B.$$

The original problem $(\mathrm{P})_B$ is called *stable* if $v_B = w_B$, and if there exists an optimal solution for the dual problem $(\mathrm{P}^*)_B$. A well-known stability criterion (cf. [2, p. 52]) is the condition

$(\mathrm{S})_B$ There exists $x_0 \in E$ and a neighborhood V of 0 in Y such that $\Phi(x_0, \cdot): V \to F$ is continuous at 0.

We derive a weaker stability criterion in Section 7.

4. Lagrange Duality

Let E, F be as in Section 2, and let G be a normed space ordered by a normal positive cone G_+. Let $\phi: E \to F \cup \{\infty\}$ and $\chi: E \to G \cup \{\infty\}$ be convex operators. We consider the optimization problem

$(\mathrm{P})_L$ minimize $\phi(x)$ subject to $\chi(x) \leqslant 0$.

The value v_L of $(\mathrm{P})_L$ is $v_L = \inf\{\phi(x): x \in E, \chi(x) \leqslant 0\}$. The dual Lagrange optimization problem $(\mathrm{P}^*)_L$ is obtained as

$(\mathrm{P}^*)_L$ maximize $\inf_x (\phi(x) + f(\chi(x)))$, $f \in \mathscr{L}(G, F)$ subject to $f \geqslant 0$.

The value of the dual problem being $w_L = \sup_{f \geqslant 0} \inf_x (\phi(x) + f(\chi(x)))$, we again have the relation

PROPOSITION 3. $-\infty \leqslant w_L \leqslant v_L$.

Indeed, for $f \in \mathscr{L}(G, F)$, $f \geqslant 0$, we find

$$\begin{aligned} \inf_x (\phi(x) + f(\chi(x))) &\leqslant \inf_{\chi(x) \leqslant 0} (\phi(x) + f(\chi(x))) \\ &\leqslant \inf_{\chi(x) \leqslant 0} \phi(x) = v_L. \end{aligned}$$

Problem $(\mathrm{P})_L$ is called *stable* if $v_L = w_L$, and if $(\mathrm{P}^*)_L$ admits an optimal solution f_0. The latter is called a *Lagrange multiplier* for problem $(\mathrm{P})_L$ in view of the relation

$$f_0(\chi(x_0)) = 0,$$

pertaining to every optimal solution x_0 of $(\mathrm{P})_L$. This corresponds with the classical (finite-dimensional) convex programming case, where Lagrange multipliers annihilate inactive constraints.

A well-known stability criterion for problem $(\mathrm{P})_L$ is Slater's condition

$(\mathrm{S})_L$ $\chi(x_0) \in -\operatorname{int} G_+$ for some $x_0 \in D(\phi)$,

which usually is combined with certain continuity assumptions on the functions ϕ, χ (cf. [7, p. 66f]). We derive a weaker stability criterion, called Rockafellar's condition, in Section 8. Also we see, then, that the continuity assumptions combined with $(\mathrm{S})_L$ are not actually needed.

5. FENCHEL DUALITY

Let E, F be as above and let $\phi: E \to F \cup \{\infty\}$ be a convex, and $\psi: E \to F \cup \{-\infty\}$ a concave operator. The primal Fenchel optimization problem is defined as

$(\mathrm{P})_F$ minimize $\phi(x) - \psi(x)$, $x \in D(\phi) \cap D(\psi)$,

its value being $v_F = \inf\{\phi(x) - \psi(x): x \in D(\phi) \cap D(\psi)\}$. Using the Young–Fenchel transforms, one states the dual problem $(\mathrm{P}^*)_F$ as

$(\mathrm{P}^*)_F$ maximize $\psi^*(f) - \phi^*(f)$, $f \in D(\phi^*) \cap D(\psi^*)$,

where ψ^* denotes the concave Young–Fenchel transform $-(-\psi)^*$.

Denoting by w_F the value of the dual problem $(\mathrm{P}^*)_F$, we have the obvious

PROPOSITION 4. $-\infty \leqslant w_F \leqslant v_F$.

This follows from the estimate

$$\begin{aligned} w_F &= \sup_f (\psi^*(f) - \phi^*(f)) \\ &= \sup_f (\inf_y (f(y) - \psi(y)) - \sup_x (f(x) - \phi(x))) \\ &\leqslant \sup_f (f(x) - \psi(x) - (f(x) - \phi(x))) \leqslant v_F. \end{aligned}$$

Again, $(\mathrm{P})_F$ is called *stable* if $w_F = v_F$ holds and a dual optimal solution exists. A sufficient condition for stability was obtained by Zowe in [12]:

$(\mathrm{S})_F$ ϕ is continuous at some $\bar{x} \in \operatorname{int} D(\phi) \cap D(\psi)$.

We obtain a weaker stability condition in the next section.

6. STABILITY FOR $(\mathrm{P})_F$

In this section we reduce the Fenchel optimization problem to the Sandwich Theorem, thereby obtaining the appropriate setting for stability.

Let $(\mathrm{P})_F$ be defined as in Section 5. Suppose $v_F \in F$; then

$$v_F + \psi(x) \leqslant \phi(x)$$

holds for all $x \in D(\phi) \cap D(\psi)$. The Sandwich problem for the convex operator ϕ and the concave operator $\psi + v_F$ now consists in finding a *continuous* affine mapping $h: E \to F$ satisfying

$$\psi + v_F \leqslant h \leqslant \phi.$$

Suppose this Sandwich problem has been solved and $h = f_0 + w$ for certain $f_0 \in \mathscr{L}(E, F)$, $w \in F$ has been found accordingly. Then f_0 actually *is* an optimal solution for $(\mathrm{P}^*)_F$, rendering problem $(\mathrm{P})_F$ stable. This follows from the estimate

$$\phi(x) - \psi(y) - v_F \geqslant h(x) - h(y) = f_0(x) - f_0(y),$$

pertaining to all $x \in D(\phi)$, $y \in D(\psi)$. Rearranging this inequality gives

$$f_0(y) - \psi(y) - (f_0(x) - \phi(x)) \geqslant v_F$$

for all $x \in D(\phi)$, $y \in D(\psi)$. So passing to the infima on the left-hand side yields the desired inequality $\psi^*(f_0) - \phi^*(f_0) \geqslant v_F$, in which equality must hold as a consequence of Proposition 4. So we are led to establish a continuous Sandwich Theorem for the operators ϕ, $\psi + v_F$.

THEOREM 5. *Let* $(\mathrm{P})_F$ *be defined as in Section 5. Then* $(\mathrm{P})_F$ *is algebraically stable provided that the condition*

$(\mathrm{R})_F$ $\;D(\phi) - D(\psi)$ *is absorbing*

is satisfied. Topological stability of $(\mathrm{P})_F$ *is guaranteed by any one of the following conditions* (1)–(4):

(1) *For every* $g \in F'$, $g \geqslant 0$ *there exists* $\bar{x} \in D(\phi) \cap D(\psi)$ *such that* $g \circ \phi: D(\phi) \to \mathbb{R}$ *is upper semicontinuous at* $\bar{x}$, $g \circ \psi: D(\psi) \to \mathbb{R}$ *is lower semicontinuous at* $\bar{x}$, *and* $(D(\phi) - \bar{x}, D(\psi) - \bar{x})$ *induces an open decomposition of* E (cf. [3, 5, 6]).

(2) *E is a separable Banach space, and there exist analytic sets C, D in E having* $C \subseteq D(\phi)$, $D \subseteq D(\psi)$ *such that* $C - D$ *is absorbing, and for every* $g \in F'$, $g \geqslant 0$, *the convex functionals* $g \circ \phi: C \to \mathbb{R}$, $-g \circ \psi: D \to \mathbb{R}$ *are Borel measurably majorized.*

(3) *E is a Banach space, and there exist CS-closed sets C, D in E having* $C \subseteq D(\phi)$, $D \subseteq D(\psi)$ *such that* $C - D$ *is absorbing, and for every* $g \in F'$, $g \geqslant 0$, *the convex functionals* $g \circ \phi: C \to \mathbb{R}$, $-g \circ \psi: D \to \mathbb{R}$ *are Borel measurably majorized.*

(4) *E is Banach,* $D(\phi) - D(\psi)$ *is absorbing, and for every* $g \in F'$, $g \geqslant 0$, *the convex functionals* $g \circ \phi$, $-g \circ \psi$ *have weakly* $\mathscr{K}$*-analytic level sets* $\{g \circ \phi \leqslant \alpha\}$, $\{g \circ \psi \geqslant \beta\}$, (*cf.* [10]).

Proof. By condition $(\mathrm{R})_F$, we certainly have $v_F < \infty$. Let us first consider the case $v_F = -\infty$. Setting $f_0 = 0$ then obviously provides a dual optimal solution satisfying $\psi^*(f_0) - \phi^*(f_0) = -\infty$. So let us now assume $v_F \in F$. Define a sublinear operator $\chi: E \to F$ by setting

$$\chi(x) = \inf\{\lambda(\phi(x) - \psi(y) - v_F): \lambda > 0, z = \lambda(x - y)\}.$$

In view of condition $(\mathrm{R})_F$, χ is actually fully defined (see [4, 12]). The Hahn–Banach Theorem therefore provides a linear mapping $f_0: E \to F$ supporting χ. Certainly, f_0 is the desired optimal solution for $(\mathrm{P}^*)_F$, rendering $(\mathrm{P})_F$ stable, resp. topologically stable, once its continuity is established.

Deriving the continuity of f_0 is possible under any one of the topological conditions (1)–(4).

First consider the case where (1) is satisfied. As E is normed and F is normally ordered, it suffices to show that $g \circ f_0$ is continuous for every $g \in F'$, $g \geqslant 0$.

Let $g \in F'$, $g \geqslant 0$ be fixed. By assumption there exists $\bar{x} \in D(\phi) \cap D(\psi)$ such that $g \circ \phi$ is upper semicontinuous at $\bar{x}$, $g \circ \psi$ is lower semicontinuous at $\bar{x}$ and $(D(\phi) - \bar{x}, D(\psi) - \bar{x})$ induces an open decomposition of E.

This means that we can find $\alpha, \beta > 0$ and a neighborhood U of 0 in E such that

$$g \circ \phi \leqslant \alpha \qquad \text{on} \quad (\bar{x} + U) \cap D(\phi),$$

$$g \circ \psi \geqslant -\beta \qquad \text{on} \quad (\bar{x} + U) \cap D(\psi).$$

We claim that $g \circ f_0$ is bounded by $\alpha + \beta$ on some neighborhood of 0 in E, from which the continuity of $g \circ f_0$ follows.

Assume the contrary. Then

$$g(f_0(z_n)) > \alpha + \beta$$

for a null-sequence (z_n) in E. Now as the pair $(D(\phi) - \bar{x}, D(\psi) - \bar{x})$ induces an open decomposition of E, we can find sequences (x_n) in $D(\phi)$, (y_n) in $D(\psi)$ such that $x_n \to \bar{x}$, $y_n \to \bar{x}$, and

$$z_n = (x_n - \bar{x}) - (y_n - \bar{x}) = x_n - y_n$$

eventually.

But $x_n, y_n \in \bar{x} + U$ eventually, hence we obtain

$$g(f_0(z_n)) \leqslant g(\chi(z_n)) \leqslant g(\phi(x_n) - \psi(y_n)) \leqslant \alpha + \beta$$

eventually, a contradiction with the choice of (z_n). This proves the claim in case (1).

Case (2) is just our Sandwich Theorem [4, Theorem 6(1)], while case (3) is covered by [6] (see also [4, Theorem 6(2)]). Finally, consider case (4). In the case $F = \mathbb{R}$, this is just the Sandwich Theorem [5, Satz 3]. But note that, in view of the fact that E is a Banach space and F is normally ordered, we again must only show that $g \circ f_0$ is continuous for fixed $g \in F'$, $g \geqslant 0$. So we are left to deal with the scalar case, to which the method in [5] applies. This ends the proof in case (4). ∎

Remark. The purely algebraic part $(\mathrm{R})_F$ of the stability criteria (1)–(4) presented in Theorem 5 was already discussed in [12]. It is sufficient to obtain an algebraic optimal solution for the dual problem $(\mathrm{P}^*)_F$.

It is worth noting that condition (1) above is in some sense minimal for the stability of problem $(\mathrm{P})_F$. More precisely, we have the following

PROPOSITION 6. *Let E be a Banach space, and let C, D be convex sets in E such that $C - D$ is absorbing and $0 \in C \cap D$. Suppose that for all convex $\phi: E \to \mathbb{R} \cup \{\infty\}$, $C \subseteq D(\phi)$ and concave $\psi: E \to \mathbb{R} \cup \{-\infty\}$, $D \subseteq D(\psi)$ such* *that $\phi \mid C$, $-\psi \mid D$ are upper semicontinuous at* 0, *the corresponding Fenchel optimization problem* $(\mathrm{P})_F$ *is* (*topologically*) *stable. Then* (C, D) *induces an open decomposition of* E.

Proof. We prove that every linear functional $f \in E^*$ such that $f \mid C, f \mid D$ are continuous at 0 (on C resp. D), is continuous on E. From this we derive that (C, D) induces an open decomposition of E (cf. [5, Section 4]).

Let $f \in E^*$ be of this kind. Then ϕ, defined by $\phi \mid C = f$, $\phi(x) = \infty$ for $x \notin C$, is convex, ψ defined by $\psi \mid D = f$, $\psi(y) = -\infty$ for $y \notin D$, is concave. By assumption, the corresponding problem $(\mathrm{P})_F$ is stable, so there exists a continuous linear $f_0 \in E'$ such that

$$\begin{aligned} \inf(\mathrm{P})_F = 0 &= \psi^*(f_0) - \phi^*(f_0) \\ &= \inf_{y \in D} (f_0(y) - f(y)) - \sup_{x \in C} (f_0(x) - f(x)). \end{aligned}$$

This implies

$$f_0(x) - f(x) \leqslant f_0(y) - f(y)$$

respectively

$$f_0(x - y) \leqslant f(x - y)$$

for all $x \in C$, $y \in D$. As $C - D$ is absorbing, every $z \in E$ may be represented as $z = \lambda(x - y)$, $\lambda > 0$, $x \in C$, $y \in D$. But this implies $f_0 \leqslant f$ on E, hence $f_0 = f$, so f is continuous. ∎

7. Stability for $(\mathrm{P})_B$

Using the results of the previous section, we now derive a stability criterion for bifunctional duality, which in a sense similar to that of Proposition 6 is weakest possible. We obtain this by reducing problem $(\mathrm{P})_B$ to an equivalent Fenchel optimization problem.

Let $\Phi: E \times Y \to F \cup \{\infty\}$ be a convex bifunction with nonempty domain, and let $(\mathrm{P})_B$ be the corresponding minimization problem. We define convex sets

$$\begin{aligned} C &= \{(y, z) \in Y \times F: \Phi(x, y) \leqslant z \text{ for some } x \in E\}, \\ D &= \{0\} \times F, \end{aligned}$$

and a convex operator $\phi: Y \times F \to F \cup \{\infty\}$, by $\phi(y, z) = z$ on C, $\phi(y, z) = \infty$ otherwise, a concave operator $\psi: Y \times F \to F \cup \{-\infty\}$ by $\psi \equiv 0$ on D, $\psi \equiv -\infty$ otherwise.

Let $(\mathrm{P})_F$ denote the Fenchel optimization problem associated with ϕ and ψ. Then we have the following relations.

PROPOSITION 7. *$v_B = v_F$, $w_B = w_F$. Moreover, optimal solutions for $(\mathrm{P})_B$ and $(\mathrm{P})_F$ respectively dual optimal solutions for $(\mathrm{P}^*)_B$ and $(\mathrm{P}^*)_F$ correspond.*

Proof. The statement concerning $v_B = v_F$ and optimal solutions for $(\mathrm{P})_B$, $(\mathrm{P})_F$ is clear from the definition of $(\mathrm{P})_F$. Let us check $w_B = w_F$. Observing that

$$\psi^*(f) = \inf_z f(0, z) < \infty$$

is possible only when $f(0, \cdot) = 0$, and that $\psi^*(f) = 0$ on $D(\psi^*)$, we obtain

$$\begin{aligned} w_F &= \sup_f \{- \sup_{(y,z)\in C} (f(y, z) - z)\} \\ &= \sup_f \inf_{(y,z)\in C} (z - f(y, 0)) \\ &= \sup_f \inf\{\Phi(x, y) - f(y, 0) : (x, y) \in E \times Y\} \\ &= w_B, \end{aligned}$$

where $\sup_f$ stands for the supremum over $D(\psi^*) \cap D(\phi^*) = \{f \in \mathscr{L}(Y \times F, F) : f(0, \cdot) = 0\} = D(\Phi^*(0, \cdot))$.

Finally, the above calculation also shows that an optimal solution f_0 for $(\mathrm{P}^*)_F$ must satisfy $f_0(0, \cdot) = 0$, and so must uniquely correspond to an optimal solution of $(\mathrm{P}^*)_B$, and vice versa. ∎

As a consequence of Proposition 7 and the stability result Theorem 5 for $(\mathrm{P})_F$, we now obtain

THEOREM 8. *Let $(\mathrm{P})_B$ defined as in Section 3. $(\mathrm{P})_B$ is algebraically stable if the following condition*

$(\mathrm{R})_B$ *the projection $p_Y(D(\Phi))$ of $D(\Phi)$ onto the Y-coordinate is absorbing*

is satisfied. Topological stability is provided by any one of the following conditions (1)–(4):

(1) *The projection of $D(\Phi)$ onto the Y-coordinate is a neighborhood of* 0 *in Y;*

(2) *E, F, Y are separable Banach spaces,* $(\mathrm{R})_B$ *holds, and Φ has Borel measurable epigraph;*

(3) *E, F, Y are Banach spaces,* $(\mathrm{R})_B$ *is satisfied, and* Φ *has CS-closed epigraph;*

(4) *F is a weakly* $\mathscr{K}$*-analytic Banach space,* $(\mathrm{R})_B$ *is valid, and* Φ *has weakly* $\mathscr{K}$*-analytic level sets* $\{\Phi \leqslant z\}$, $z \in F$.

Proof. Condition $(\mathrm{R})_B$ means that for every $y \in Y$ there exist $\lambda > 0$ and $x \in E$ such that $\Phi(x, \lambda y) < \infty$. So $(\mathrm{R})_B$ translates into condition $(\mathrm{R})_F$ for the associated Fenchel optimization problem $(\mathrm{P})_F$, i.e., the projection $p_Y(C)$ is absorbing in Y. Hence the algebraic part of Theorem 5 applies. Concerning the topological part it suffices to check that (1)–(4) above correspond with (1)–(4) from Theorem 5 when applied to the special problem $(\mathrm{P})_F$ associated with $(\mathrm{P})_B$.

First consider case (1). This condition is just the statement that (C, D) induces an open decomposition of $E \times Y$. Since ϕ on C, ψ on D are obviously continuous, Theorem 5, part (1), applies and gives the result.

Next consider case (2). Here $\mathrm{Epi}(\phi) = \{(x, y, z) : z \geqslant \Phi(x, y)\}$ is Borel, hence its projection onto the coordinates (y, z) is an analytic set, $E \times Y \times F$ being a *separable* Banach space. But this is just the set C. Since $D = \{0\} \times F$ is analytic and ϕ, ψ are continuous on C, D, Theorem 6, part (2), applies.

For part (3) observe that the CS-closedness of the epigraph $\mathrm{Epi}(\Phi)$ implies that its image C under the projection $(x, y, z) \to (y, z)$ is a pseudo-complete set in the sense of [6]. So this verson follows from the slightly more general statement of the Sandwich Theorem in [6].

Finally, for part (4), observe that the functions ϕ, ψ have weakly $\mathscr{K}$-analytic level sets. For ψ this is a consequence of the weak $\mathscr{K}$-analyticity of F. For ϕ observe that $\{\phi \leqslant z\}$ is just the image of

$$\mathrm{Epi}(\Phi) \cap (E \times Y \times \{z\})$$

under the projection $(x, y, z) \to (y, z)$, hence is weakly $\mathscr{K}$-analytic. This ends the proof of Theorem 8. ∎

Remark. Note that the stability condition $(\mathrm{S})_B$ for $(\mathrm{P})_B$ presented in Section 3 clearly implies condition (1) of Theorem 8.

8. Stability for $(\mathrm{P})_L$

Let E, F, G, ϕ, χ be defined as in Section 4, and let $(\mathrm{P})_L$ be the corresponding Lagrange minimization problem. We define a convex bifunction $\Phi : E \times G \to F \cup \{\infty\}$ by

$$\Phi(x, y) = \begin{cases} \phi(x), & \text{if } \chi(x) \leqslant y, \\ \infty, & \text{otherwise.} \end{cases}$$

Let $(\mathrm{P})_B$ be the bifunctional optimization problem associated with Φ. Then we have the following

PROPOSITION 9. *$v_L = v_B$, $w_L = w_B$. Moreover, $x_0 \in E$ is an optimal solution for $(\mathrm{P})_L$ if and only if it is optimal for $(\mathrm{P})_B$. Optimal solutions f_0 for $(\mathrm{P}^*)_L$ and g_0 for $(\mathrm{P}^*)_B$ are in 1–1 correspondence via the formula $f_0(y) = g_0(x, y)$.*

Proof. The statement concerning $v_L = v_B$ and the coincidence of the optimal solutions of $(\mathrm{P})_L$ and $(\mathrm{P})_B$ are clear from the above definition of Φ. We then prove $w_L = w_B$. By definition of $(\mathrm{P}^*)_L$, we have

$$\begin{aligned} w_L &= \sup_{f \geqslant 0} \inf_x (\phi(x) + f(\chi(x))) \\ &= \sup_{f \geqslant 0} \inf_{\chi(x) \leqslant y} [\Phi(x, y) + f(y)] \\ &= \sup_{f \leqslant 0} \inf_{x, y} [\Phi(x, y) - f(y)]. \end{aligned}$$

Here the last equality comes from the fact that elements x, y for which $\chi(x) \leqslant y$ is *not* satisfied, give $\Phi(x, y) = \infty$, and so do not contribute to the infimum. But now the expression on the right-hand side is just $-\Phi^*(0, f)$, where $(0, f)$ stands for the operator $(x, y) \to f(y)$ on $E \times G$. It remains to prove

$$\sup_{f \leqslant 0} \{-\Phi^*(0, f)\} = w_B.$$

But this follows from the fact that elements $f \in \mathscr{L}(G, F)$ having $f \nleqslant 0$ do *not* contribute to the supremum here. Indeed, $D(\Phi^*(0, \cdot))$ consists of those $f \in \mathscr{L}(G, F)$ for which

$$u_f = \sup_{\chi(x) \leqslant y} \{f(y) - \phi(x)\} < \infty.$$

Fixing $\bar{x} \in D(\phi) \cap D(\chi)$, we find that $\chi(\bar{x}) \leqslant \chi(\bar{x}) + z =: y$ for $z \geqslant 0$ in G, so

$$f(z) \leqslant u_f + \phi(\bar{x}) - f(\chi(\bar{x}))$$

for all $z \geqslant 0$ in G. But clearly this is possible only in the case where $f \leqslant 0$. This establishes $w_L = w_B$.

The above calculation also shows that optimal solutions of $(\mathrm{P}^*)_L$, $(\mathrm{P}^*)_B$ are in 1–1 correspondence in the way stated. ∎

Combining Proposition 9 with the stability Theorem 8 for bifunctional duality, we obtain stability criteria for $(\mathrm{P})_L$.

THEOREM 10. *Let* $(\mathrm{P})_L$ *be defined as in Section* 4. *Suppose the following condition* $(\mathrm{R})_L$, *called Rockafellar's condition, is satisfied*:

$(\mathrm{R})_L$ *for every* $y \in F$ *there exist* $\lambda > 0$ *and* $x \in D(\phi)$ *having* $\chi(x) \leqslant \lambda y$.

Then problem $(\mathrm{P})_L$ *is algebraically stable. Topological stability is provided by any one of the following conditions*:

(1) *The projection of* $(D(\phi) \times G) \cap \mathrm{Epi}(\chi)$ *onto the G-coordinate is a neighborhood of* 0 *in G.*

(2) *E, F, G are separable Banach spaces,* $(\mathrm{R})_L$ *is satisfied, and* ϕ, χ *have Borel measurable epigraphs.*

(3) *E, F, G are Banach spaces,* $(\mathrm{R})_L$ *is satisfied, and* ϕ, χ *have CS-closed epigraphs.*

(4) *E, F, G are weakly* $\mathscr{K}$*-analytic Banach spaces,* $(\mathrm{R})_L$ *is satisfied, and* ϕ, χ *have weakly* $\mathscr{K}$*-analytic epigraphs.*

Proof. The algebraic condition $(\mathrm{R})_L$ translates into $(\mathrm{R})_B$ for the associated bifunctional problem $(\mathrm{P})_B$. So the algebraic part of the statement follows from the algebraic part of Theorem 8. Concerning the topological part, we must show that conditions (1)–(4) translate into conditions (1)–(4) from Theorem 8.

Condition (1) above translates to (1) from Theorem 8, because of $D(\Phi) = (D(\phi) \times G) \cap \mathrm{Epi}(\chi)$. Concerning condition (2), observe that

$$\mathrm{Epi}(\Phi) = (\mathrm{Epi}(\phi) \times G) \cap (\mathrm{Epi}(\chi) \times F);$$

hence $\mathrm{Epi}(\Phi)$ is analytic if $\mathrm{Epi}(\phi)$, $\mathrm{Epi}(\chi)$, and G, F are. The same equality also shows that $\mathrm{Epi}(\Phi)$ is *CS*-closed, if $\mathrm{Epi}(\phi)$, $\mathrm{Epi}(\chi)$ are *CS*-closed. This means that Theorem 8, part (3), applies when we translate problem $(\mathrm{P})_L$ into the corresponding bifunctional problem.

Finally, the above equality also shows that in the case of condition (4), $\mathrm{Epi}(\Phi)$ is weakly $\mathscr{K}$-analytic, as $\mathrm{Epi}(\phi)$, $\mathrm{Epi}(\chi)$ and F, G are. So Theorem 8, part (4), applies here. ∎

Remarks. (1) The stability criterion (1) should be compared with the criteria in [7, p. 63 ff]. Essentially, it was first used by Rockafellar [8] in the form $(\mathrm{R})_L$, under additional continuity assumptions on ϕ, χ. Note that its advantage over the Slater condition $(\mathrm{S})_L$ (cf. Section 4) is that the positive cone G_+ need not have interior points. For instance, condition $(\mathrm{R})_L$ also permits a treatment of problem $(\mathrm{P})_L$ with affine equality constraints $\chi(x) = 0$, for in this case, the positive cone G_+ has just to be chosen as $G_+ = \{0\}$. Although G_+ does not have interior points then, $(\mathrm{R})_L$ will be satisfied in this case when the affine operator χ is onto. In particular, our formulation of problem $(P)_L$ also covers the case of mixed constraints (cf. [7, p. 66 ff]); i.e., $(R)_L$ is still a reasonable condition for stability of $(P)_L$ then.

(2) The translation of $(P)_L$ into $(P)_B$ presented in this section also shows that $(S)_L$ translates into $(S)_B$. So $(S)_L$ is a stability criterion for $(P)_L$ without any additional continuity assumptions on ϕ resp. χ. Again we refer to the results in [7, p. 63 ff], where stability of $(P)_L$ is deduced from $(S)_L$ plus additional requirements on ϕ, χ, using a different technique.

9. Conclusion

Closing the circel started in Section 6, we could translate the Fenchel optimization problem $(P)_F$ into an associated Lagrange optimization problem $(P)_L$, such that dual programs $(P^*)_F$ and $(P^*)_L$ correspond. This would also translate the stability criteria $(R)_F$ (resp. (1)–(4) for $(P)_F$) into $(R)_L$ (resp. (1)–(4) for $(P)_L$.) Starting with $(P)_F$, defined by ϕ, ψ as in Section 5, we just had to define $\tilde{\phi}(x, y) = \phi(x) - \psi(y)$, $\tilde{\chi}(x, y) = y - x$, then the problem $(P)_L$ of minimizing $\tilde{\phi}(x, y)$ subject to the affine equality constraint $\tilde{\chi}(x, y) = 0$ would suit. We refer to [7, p. 82 ff] for the treatment of the case $F = \mathbb{R}$.

Clearly the stability criteria $(S)_F$, $(S)_B$, $(S)_L$ imply the weaker $(R)_F$ (resp. (1) for $(P)_F$), $(R)_B$ (resp. (1) for $(P)_B$) and $(R)_L$ (resp. (1) for $(P)_L$).

But $(S)_F$, $(S)_B$, $(S)_L$ are *not* mutually equivalent. While $(S)_L$ implies $(S)_B$ without any continuity assumptions on ϕ, χ, condition $(S)_B$ translates into $(S)_F$ only in the case where F has an order-unit (see Section 7). Finally, the translation of $(P)_F$ into $(P)_L$ described above has no effect at all on conditions $(S)_F$, $(S)_L$, for $(S)_L$ is not appropriate when affine equality constraints are involved. Nevertheless, translating $(P)_F$ into $(P)_L$ *has* the desired effect on $(R)_F$, $(R)_L$.

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