Banach-Mazur game and open mapping theorem

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Abstract

We discuss a variant of the Banach-Mazur game which has applications to topological open mapping and closed graph theorems.

Key Words

Open mapping theorem · Closed graph theorem · Banach-Mazur game · Baire space

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1 Introduction

A Hausdorff topological space E is called a *B-space*, respectively, a *B_r-space*, if every continuous, nearly open surjection, respectively bijection, f from E onto an arbitrary Hausdorff space F is open. This definition honors V. Pták's famous open mapping theorem [31, 32, 17] for linear operators between locally convex vector spaces, where he calls spaces E satisfying the open mapping theorem *B-complete*, respectively, *B_r-complete*. The concept has been extended to other categories, e.g. Husain [13] calls a separated topological group G a *B-group* (a *B_r-group*) if every continuous nearly open surjective (bijective) homomorphism f from G into any separated topological group H is open. In the framework of topological vector spaces, B- and B_r-completeness have been studied in [1].

Weston [36] proved that completely metrizable spaces are B_r-spaces, and Byczkowski and Pol [2] extended this to Čech-complete spaces. In [24, 27] almost Čech complete spaces were shown to be B_r-spaces, while locally compact spaces and Lindelöff P-spaces are B-spaces [24]. Clearly every B-space is a B_r-space, while the converse is not true.

A natural question is whether, or in what sense, B- and B_r-spaces have to be *complete*. Pták's terminology points to the fact that locally convex spaces satisfying the open mapping theorem are indeed *complete*. But this is already different in the framework of topological groups, where B- and B_r-groups may be incomplete and even of the first category. Since H-minimal topological spaces are B_r-spaces, and since Herrlich [11] exhibits a first category H-minimal space, the topological case seems at first to resemble the situation in groups. However, within metrizable spaces the situation is different, as all zero-dimensional and all sub-orderable metrizable B_r-spaces are Baire [27]. It is still an open question whether *all* metrizable B_r-spaces must be Baire.

What is know is that there exist metrizable B_r-spaces which do not contain any dense completely metrizable subspace. Yet, all presently known constructions of metrizable

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B_r-spaces seem to require a somewhat strengthened form of Baire category. This is in line with the fact that metrizability and Baire category *alone* do *not* suffice to make a B_r-space.

Presently we elucidate this situation a little further by discussing a novel variant of the Banach-Mazur game, where β-defavorable spaces are B_r-spaces. This new variant is of interest in itself, as the β-defavorable case implies Baire category not just of E, but of $E \times E$, while it still allows $E \times E \times E$ to be of first category, and E to be barely Baire. The α-favorable case of the new game coincides with the α-favorable case of the classical Banach-Mazur game, but otherwise differences occur to the effect that the gap between α-favorability and β-favorability narrows. We even identify classes of spaces where the game is determined.

It turns out that it is reasonable to consider, along with B- and B_r-spaces, a third intermediate concept. We call a Hausdorff space E a *B_q-space*, if every continuous nearly open surjection f onto any Hausdorff space F is a quotient map. In those categories where quotient maps are by default open this offers nothing new, but B- and B_q-spaces differ in the topological case. In fact, on closer regard, topological B-spaces turn out a far more restrictive class than one would expect from the situation in groups or topological vector spaces. Here B_q-spaces are a better behaved and sufficiently rich class to warrant further independent study.

The structure of the paper is as follows. After recalling the Banach-Mazur game in Section 3, we introduce the novel variant in Section 4. Using the new game we obtain our first open mapping theorem in Section 5, followed by a second open mapping theorem in Section 6 based on the strong variant of the game. The quotient mapping theorem is addressed in Section 7. In Section 8 we resume the study of the new game and exhibit classes of spaces where it is determined. Sections 9,10 on the other hand use the barely Baire spaces of [9] to show that the new game is not determined in general. In Section 11 we have a glimpse at the Michael and strong Choquet games and show that β-defavorability for those does not imply β-defavorability for our new game. In the final Section 12 we prove a variant of the closed graph theorem, where again two-person game characterizations play a crucial role.

2 Preparations

A mapping $f : E \to F$ is *nearly open* if for every $x \in E$ and every neighborhood U of x the set $\overline{f(U)}$ is a neighborhood of $f(x)$ in F. Analogously, $f : E \to F$ is *nearly continuous* if for every $x \in E$ and every neighborhood V of $f(x)$ in F, the set $\overline{f^{-1}(V)}$ is a neighborhood of x.

Let E be a Hausdorff space. A pair (T, ϕ) consisting of a tree $T = (T, \leqslant_T)$ of height ω and a mapping ϕ from T to the non-empty open subsets of E is called a *web* on E if

- (w_1) $\{\phi(t) : t \in T\}$ is a pseudo-base of E, i.e., every non-empty open $U \subset E$ contains some $\phi(t)$.
- (w_2) For every $t \in T$ the set $\{\phi(s) : t <_T s\}$ is a pseudo-base of $\phi(t)$, i.e., every non-empty open $V \subset \phi(t)$ contains some $\phi(s)$ with $t <_T s$.

Definition (Completeness)**.** A Hausdorff space E is *p-complete* if it admits a web (T, ϕ) with the following property:

(p) For every cofinal branch $b \subset T$ the intersection $\bigcap_{t\in b} \phi(t) \neq \emptyset$ is non-empty.

The space E is *c-complete* if it admits a web (T, ϕ) with the following property:

(c) Every filter $\mathscr{F}$ containing all elements $\phi(t)$, $t \in b$, of a cofinal branch $b \subset T$ has a cluster point in $\bigcap_{t\in b} \phi(t)$.

Pseudo-complete space in the sense of Oxtoby [29] are p-complete, and p-completeness is also known as weak α-favorability for a player with perfect information [37]. Almost Čech-complete space are c-complete, and coincide with the almost-complete spaces of [19]. Clearly c-completeness implies p-completeness. Webs appear first in [7, 24], and almost-sieves [19] are closely related. Our terminology generally follows [8], but we assume all spaces to be Hausdorff.

3 Banach-Mazur game

The Banach-Mazur game in a topological space E is played by two players α and β in the following way. Player β starts and chooses a non-empty open set V_1. Then player α chooses a non-empty open subset $U_1 \subset V_1$. Next player β chooses a nonempty open $V_2 \subset U_1$, and then α a non-empty open $U_2 \subset V_2$, etc. Player α wins when $\bigcap_{n=1}^{\infty} U_n \neq \emptyset$, while player β wins when $\bigcap_{n=1}^{\infty} U_n = \emptyset$.

A strategy for player β is a mapping, for simplicity also noted β, which for every sequence of non-empty open sets $V_1, U_1, V_2, U_2, \dots, V_k, U_k$ of even length $2k$ chooses a non-empty open set $V_{k+1} = \beta(V_1, U_1, \dots, V_k, U_k) \subset U_k$. This includes the sequence of length 0, where $\beta(\emptyset) = V_1 \neq \emptyset$. Similarly, a strategy for player α is a mapping, noted α, which for every sequence $V_1, U_1, \dots, V_k$ of non-empty open sets of odd length $2k-1$ chooses a non-empty open $U_k = \alpha(V_1, U_1, \dots, V_k) \subset V_k$. The play of α against β is the sequence $V_1 \supset U_1 \supset V_2 \supset U_2 \dots$ satisfying $V_1 = \beta(\emptyset)$, $U_1 = \alpha(V_1)$, $V_2 = \beta(V_1, U_1)$, $U_2 = \alpha(V_1, U_1, V_2)$, etc. We call this the BM-game.

Theorem 1. *E is a Baire space if and only if for every strategy β in the* BM*-game there exists a strategy α beating it.* □

The class of spaces E where α has a winning strategy has been characterized by White [37], who called them weakly α-favorable for a player with perfect information. These are precisely the p-complete spaces above.

It is custom to call a space α-favorable if player α has a winning strategy, and α-defavorable if it has not. The terms β-favorable and β-defavorable are used in the same sense. This terminology will also be used for variants of the BM-game, and all variants will use perfect memory.

4 Tandem Banach-Mazur game

Now we introduce a variant of the Banach-Mazur game. We have two players α' and β'. Player β' first chooses a non-empty open V_1, to which player α' responds with a non-empty open $V_1' \subset V_1$. The second move of player β' is a non-empty open W_1, to which player α' responds with a non-empty open $W_1' \subset W_1$. Then player β' switches back to the V-side and chooses a non-empty open $V_2 \subset V_1'$, to which α' responds with a non-empty open $V_2' \subset V_2$. Then β' chooses $W_2 \subset W_1'$, and α' responds by $W_2' \subset W_2$, and so on. Players alternate between the V-side and the W-side, playing one move each on one side, before switching to the other side and playing one move each there, etc. The play therefore generates two nested sequences $V_1 \supset V_1' \supset V_2 \supset V_2' \supset \dots$ and $W_1 \supset W_1' \supset W_2 \supset W_2' \supset \dots$, but arranged in the following meandering order

$$
\begin{array}{ccccccccc}
\beta' & & \alpha' & & \beta' & & \alpha' & & \beta' \\
\hline
V_1 & \to & V_1' & & V_2 & \to & V_2' & & \dots \\
 & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
 & & W_1 & \to & W_1' & & W_2 & \to & W_2' \\
\hline
 & & \beta' & & \alpha' & & \beta' & & \alpha'
\end{array}
$$

We assume that both players have full information of the past from both boards as time follows the arrows. Player α' wins when both $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$ *and* $\bigcap_{n=1}^{\infty} W_n \neq \emptyset$, while player β' wins as soon as at least one of these intersections is empty. We call this the *tandem Banach-Mazur game*, or for short, the BM$'$-game.

Proposition 1. *Player α has a winning strategy in the* BM*-game if and only if player α' has a winning strategy in the* BM$'$*-game.*

Proof: 1) Suppose α is winning. We construct a winning strategy α' by forgetting about the intertwined nature of the BM$'$-game and by reacting to the moves of β' on the V- and W-board as if those were two non-cooperating players β_V, β_W in the role of β in the standard BM-game. Since α wins against all strategies, it wins against these two, hence α' is winning as well.

2) Conversely, suppose α' is winning. We construct a winning strategy α. Observe that players β' may choose their moves such that $W_i \subset V_i'$, and also, $V_{i+1} \subset W_i'$, in which event the play in the BM$'$-game will consist of one single nested sequence and will correspond to a play in the BM-game. Therefore we may reinterpret β as such a willful strategy β'. The moves of the winning strategy α' on the V- and W-board may then be translated back to moves of a strategy α in the BM-game, which is winning since α' is. $\square$

Differences between the two games are expected when no winning strategy α exists. Let us observe that in order to win player β' has only to arrange for one of the nested sequences $V_i \supset V_{i+1}$ or $W_i \supset W_{i+1}$ to have empty intersection. It is therefore clear that a winning strategy β in the standard BM-game gives rise to a winning strategy for β' in the BM$'$-game, by just concentrating on winning on one of the boards.

Proposition 2. *Suppose β has a winning strategy in the* BM*-game, then β' has a winning strategy in the* BM$'$*-game.*

We will see later (Remarks 2 and 14) that the converse is not true, i.e., β' may have a winning strategy, while β has none.

An easy consequence of Proposition 2 is that if in E the BM$'$-game is β'-defavorable, then E is a Baire space. But we have the following stronger

Theorem 2. *Suppose for every strategy β' in the* BM$'$*-game on E there exists a strategy α' which wins against β'. Then $E \times E$ is a Baire space.*

Proof: We look at the Banach-Mazur game in the space $E \times E$ with players α, β, and associate with them players α', β' in the BM$'$-game on E.

We may without loss of generality assume that β plays with non-empty open boxes. Suppose $\beta(\emptyset) = V_1 \times W_1$. We interpret V_1 as the first move of player β' in the BM$'$-game on E on the V-board, to which player α' responds by $V_1' \subset V_1$ nonempty. Normally, in the BM$'$-game players β' have now the possibility to choose their move W_1 by taking V_1, V_1' into account. But we take W_1 as the second component of the first move of β and let this be the second move of β' in the BM$'$-game, now on the W-board. More formally, with p_1, p_2 the projections on first and second coordinate, $\beta'(p_1(\beta(\emptyset)), V_1') = p_2(\beta(\emptyset))$ independently of the choice $V_1' \subset V_1 = p_1(\beta(\emptyset))$. In other words, β' wastes the option to take V_1', V_1 into account. Player α' reacts by choosing $W_1' \subset W_1$ based on all the previous information. Then we re-interpret $V_1' \times W_1'$ as the move of player α in $E \times E$, i.e., $\alpha(V_1 \times W_1) = V_1' \times W_1' = \alpha'(V_1) \times \alpha'(V_1, \alpha'(V_1), W_1)$. We repeat this procedure in the following sweeps.

Once this wasteful strategy β' in the BM$'$-game is defined, by hypothesis there exists α' beating β'. The play so obtained may now be read as a play $V_1 \times W_1 \supset V_1' \times W_1' \supset V_2 \times W_2 \supset \dots$ between α and β in the BM-game on $E \times E$. Since α' wins, we have $\bigcap_{i=1}^\infty V_i \neq \emptyset$ and $\bigcap_{i=1}^\infty W_i \neq \emptyset$, hence of course $\bigcap_{i=1}^\infty V_i \times W_i \neq \emptyset$, hence α beats β. By Theorem 1, $E \times E$ is a Baire space. $\square$

Remark 1. This construction generates a BM-strategy α on $E \times E$ winning against β, where α plays with open boxes in $E \times E$. It is not clear whether such a strategy exists when it is only known that $E \times E$ is Baire. We can assume that β plays with open boxes, but it is by no means clear whether this can be arranged for the α beating it.

Definition (τ-Baire space)**.** A topological space E in which for every strategy β' in the BM$'$-game there exists a strategy α' beating it is called a *τ-Baire space.*

Remark 2. It is well-known that there exist Baire spaces E whose square $E \times E$ is no longer Baire (see [9] and Section 10), and such a space is Baire but not τ-Baire. In Remark 14 we will see that even when $E \times E$ is Baire, this still does not mean that E is τ-Baire.

5 First open mapping theorem

A Hausdorff space F is called a δ-space if it admits a web (T, ϕ) with the following property:

(d) For every cofinal branch $b \subset T$ the intersection $\bigcap_{t \in b} \phi(t)$ contains at most one point.

Recall that a Hausdorff space is called *semi-regular* if the family of regular-open sets is a basis for the topology; [8, p. 58]. We are now ready to prove our first open mapping theorem:

Theorem 3. *Let E be a semi-regular τ-Baire space, F a δ-space. If $f : E \to F$ is a continuous nearly open bijection, then f is open.*

Proof: Let (T, ϕ) be a web on F satisfying (d). We have to show that f is open. Let $x \in E$ and U a neighborhood of x. Using semi-regularity, choose an open neighborhood V of x with $\overline{V}^\circ \subset U$. It suffices to prove $\overline{f(V)}^\circ \subset f(U)$, as this will show openness at x.

Let $y \in \overline{f(V)}^\circ$, $y = f(z)$. We will show $z \in \overline{V}^\circ$, as this gives $z \in U$. By continuity of f there exists an open O with $z \in O$ and $f(O) \subset \overline{f(V)}^\circ$. Proving $O \subset \overline{V}$ will now be sufficient. Take $w \in O$ and an arbitrary open neighborhood W of w, which may be assumed to satisfy $W \subset O$. It remains to prove $V \cap W \neq \emptyset$, as this will show $w \in \overline{V}$.

We define a strategy β' in the BM$'$-game on E. We start with the definition of $\beta'(\emptyset)$. Since $\{\phi(t) : t \in T\}$ is a pseudo-base of F by (w_1), the set $D_1 = \bigcup\{\phi(t) : t \in T, \phi(t) \subset \overline{f(V)}^\circ\}$ is open dense in $\overline{f(V)}^\circ$. On the other hand we have $f(W) \subset \overline{f(V)}^\circ$. Now we apply Lemma 1 (below) with the choices $G = D_1$, $H = \overline{f(V)}^\circ$, $O = V$, $U = W$. We conclude that $D_1 \cap \overline{f(W)}^\circ \cap f(V) \neq \emptyset$. By the definition of D_1 we can pick $t_1 \in T$ with $\phi(t_1) \subset \overline{f(W)}^\circ$ and $x_1 \in V$ satisfying $f(x_1) \in \phi(t_1)$. Now by continuity find an open V_1 with $x_1 \in V_1 \subset V$ and $f(V_1) \subset \phi(t_1)$. Our move is $\beta'(\emptyset) = V_1$.

Let $V_1' \subset V_1$ be a potential response of player α'. We have to define $\beta'(V_1, V_1')$. The set $D_2 = \bigcup\{\phi(t) : t \in T, t_1 <_T t\}$ is open dense in $\phi(t_1)$ by (w_2), while from $V_1' \subset V_1$ we obtain $f(V_1') \subset \phi(t_1) \subset \overline{f(W)}^\circ$. We may therefore apply Lemma 1 with the choices $G = D_2$, $H = \phi(t_1)$, $U = V_1'$ and $O = W$. The conclusion is that $\overline{f(V_1')}^\circ$ intersects $f(W) \cap D_2$. By the definition of D_2 choose $t_2 \in T$ with $t_1 <_T t_2$ and $\phi(t_2) \subset \overline{f(V_1')}^\circ$, together with $y_1 \in W$ such that $f(y_1) \in \phi(t_2)$. By continuity find W_1 open with $y_1 \in W_1 \subset W$ satisfying $f(W_1) \subset \phi(t_2)$, and let $\beta'(V_1, V_1') = W_1$ be our move.

Now let $W_1' \subset W_1$ be nonempty open, then we have to define $\beta'(V_1, V_1', W_1, W_1')$. Note that $D_3 := \bigcup\{\phi(t) : t_2 <_T t\}$ is dense in $\phi(t_2)$, while $f(W_1') \subset \phi(t_2) \subset \overline{f(V_1')}^\circ$. This allows us to apply Lemma 1 with the choices $H = \phi(t_2)$, $G = D_3$, $U = W_1'$ and $O = V_1'$. We conclude that $\overline{f(W_1')}^\circ$ intersects $f(V_1') \cap D_3$. We can therefore pick $t_3 \in T$ with $t_2 <_T t_3$ satisfying $\phi(t_3) \subset \overline{f(W_1')}^\circ$ and $x_2 \in V_1'$ with $f(x_2) \in \phi(t_3)$. Pick an open V_2 with $x_2 \in V_2 \subset V_1'$ satisfying $f(V_2) \subset \phi(t_3)$, and put $\beta'(V_1, V_1', W_1, W_1') = V_2$.

Let $V_2' \subset V_2$ be nonempty open. We have to define $\beta'(V_1, V_1', W_1, W_1', V_2, V_2')$. The set $D_4 = \bigcup\{\phi(t) : t_3 <_T t\}$ is dense in $\phi(t_3)$, while $f(V_2') \subset f(V_2) \subset \phi(t_3) \subset \overline{f(W_1')}^\circ$. So we apply Lemma 1 with the choices $G = D_4$, $H = \phi(t_3)$, $O = W_1'$, $U = V_2'$. The consequence is that $\overline{f(V_2')}^\circ$ intersects $f(W_1') \cap D_4$. We pick $t_4 \in T$, $t_3 <_T t_4$ with $\phi(t_4) \subset \overline{f(V_2')}^\circ$ and $y_2 \in W_1'$ with $f(y_2) \in \phi(t_4)$. Then we choose an open set W_2 with $y_2 \in W_2 \subset W_1'$ such that $f(W_2) \subset \phi(t_4)$, and let $\beta'(V_1, V_1', W_1, W_1', V_2, V_2') = W_2$ be our move.

Continuing in this way defines a strategy β' in the BM$'$-game, and since E is τ-Baire, we find a strategy α' winning against β'. Let $V_1, V_1', W_1, W_1', V_2, V_2', W_2, W_2', \dots$ be their play. Then we find $v \in \bigcap_{i=1}^\infty V_i \neq \emptyset$ and $w \in \bigcap_{i=1}^\infty W_i \neq \emptyset$. In particular, $v \in V$ and $w \in W$. At the same time $f(v) \in f(V_i) \subset \phi(t_{2i-1})$ and $f(w) \in f(W_i) \subset \phi(t_{2i})$. Hence $f(v), f(w) \in \bigcap_{i=1}^\infty \phi(t_i)$, and since $t_1 <_T t_2 <_T t_3 <_T \dots$ is a cofinal branch in T, we get $f(v) = f(w)$ from (d). With f being injective, this gives $v = w$, hence $V \cap W \neq \emptyset$. That ends the proof. $\square$

Lemma 1. *Let $f : E \to F$ be continuous and nearly open. Let $O, U \subset E$ be open and $G, H \subset F$ open with $G \subset H \subset \overline{G}$. Suppose $f(U) \subset H \subset \overline{f(O)}^\circ$. Then $G \cap \overline{f(U)}^\circ \cap f(O) \neq \emptyset$.*

Proof: Observe that $G \cap \overline{f(U)}^\circ \neq \emptyset$. For had we $G \cap \overline{f(U)}^\circ = \emptyset$, then also $\overline{G} \cap \overline{f(U)}^\circ = \emptyset$, hence $H \cap \overline{f(U)}^\circ = \emptyset$ due to $H \subset \overline{G}$. But due to near openness that contradicts the assumption $f(U) \subset H$. Hence $G \cap \overline{f(U)}^\circ \neq \emptyset$.

From $f(U) \subset \overline{f(O)}^\circ$ follows $\overline{f(U)}^\circ \subset \overline{f(O)}^\circ$, and since $G \subset \overline{f(O)}^\circ$ anyway, the open set $G \cap \overline{f(U)}^\circ$ is contained in $\overline{f(O)}$. As it is nonempty by the above, it must intersect the dense subset $f(O)$ of $\overline{f(O)}$. $\square$

Following Jayne and Rogers [14], a topological space F is *fragmentable* if there exists a metric d on F such that for every $\epsilon > 0$ and every nonempty set $X \subset F$ there exists an open set U in F such that $Y = X \cap U$ is nonempty and has diameter $\leq \epsilon$ with respect to d. For a game-theoretic characterization of fragmentability see [16].

Following Čoban et al. [4, 5] a space F is *open-fragmentable* if the above condition applies to open sets $X \subset F$ only, in which case Y is also open. The authors call this a *fos*-space and obtain a related game-theoretic characterization, namely, using plays $V_1 \supset W_1 \supset V_2 \supset W_2 \supset \dots$ where player α wins if $\bigcap_{i=1}^{\infty} W_i$ consists of at most one point. Then F is open-fragmentable iff player α has a winning strategy in this game called the FO-game. See also [35] for further properties of this class.

Lemma 2. *A space F is open-fragmentable iff it is a δ-space.*

Proof: 1) Suppose F is open-fragmentable. Define a tree T of height ω as follows. The elements t of T are finite sequences of nonempty open sets $t = (U_1, U_2, \dots, U_n)$ with $U_1 \supset U_2 \supset \dots \supset U_n$ satisfying $\mathrm{diam}(U_i) \leq 1/i$. The order relation is extension of sequences. The mapping ϕ is $(U_1, \dots, U_n) \to U_n$. It is clear that if $t_1 <_T t_2 <_T \dots$ is a cofinal branch, then this gives rise to a nested sequence $U_1 \supset U_2 \supset \dots$ with $\bigcap_{i=1}^{\infty} U_i$ containing at most one point, hence property (d) is guaranteed. We still have to prove that (T, ϕ) is a web. Let us check property (w_2). Let $t = (U_1, \dots, U_n) \in T$. For every nonempty open $O \subset U_n$ there exists a nonempty open $U_{n+1} \subset O$ with $\mathrm{diam}(U_{n+1}) \leq 1/(n+1)$. Hence $\{U_{n+1} : (U_1, \dots, U_n, U_{n+1}) \in T\}$ is a pseudo-base of the set $U_n = \phi(t)$. The argument for (w_1) is similar.

2) For the converse, a web (T, ϕ) satisfying $(w_1), (w_2), (d)$ may obviously be used to define a winning strategy for α in the FO-game. Construction of a metric which fragments open sets then follows as in [5]. $\square$

6 Second open mapping theorem

Now we consider the strong version of the BM$'$-game, where player α' wins the play $(V_1, V_1', W_1, W_1', V_2, V_2', \dots)$ *strongly* against β' if every filter $\mathscr{F}_V$ with $V_i \in \mathscr{F}_V$ for all i has a cluster point in $\bigcap_{i=1}^{\infty} V_i$, and every filter $\mathscr{F}_W$ with $W_i \in \mathscr{F}_W$ for all i has a cluster point in $\bigcap_{i=1}^{\infty} W_i$. If for every strategy β' there exists a strategy α' winning strongly against β', then we call E a τ^*-Baire space. The c-complete spaces of Section 2 are those where players α, or α', have a strong winning strategy, so c-complete spaces are τ^*-Baire.

Theorem 4. *Every semi-regular τ^*-Baire space E is a B_r-space.*

Proof: Consider a continuous nearly open bijection $f : E \to F$. We have to show that f is open. Let $x \in E$ and U a neighborhood of x. Using semi-regularity, choose an open neighborhood V of x with $\overline{V}^{\circ} \subset U$. It suffices to prove $\overline{f(V)}^{\circ} \subset f(U)$, as this will show openness at x.

Let $y \in \overline{f(V)}^{\circ}$, $y = f(z)$. We will show $z \in \overline{V}^{\circ}$, as this gives $z \in U$. By continuity of f there exists an open O with $z \in O$ and $f(O) \subset \overline{f(V)}^{\circ}$. Proving $O \subset \overline{V}$ will now be sufficient. Take $w \in O$ and an arbitrary open neighborhood W of w, which may be assumed to satisfy $W \subset O$. It remains to prove $V \cap W \neq \emptyset$, as this will show $z \in \overline{V}$.

We are going to define a strategy β' in the BM$'$-game on E. To start, observe that $\overline{f(W)}^{\circ} \cap f(V) \neq \emptyset$, because $f(V)$ is dense in $\overline{f(V)}^{\circ}$, and $\overline{f(W)}^{\circ}$ intersects $\overline{f(V)}^{\circ}$, given that $f(W) \subset f(O) \subset \overline{f(V)}^{\circ}$. Since $\overline{f(W)}^{\circ} \cap f(V) \neq \emptyset$, we can find a nonempty open set V_1 with $f(V_1) \subset \overline{f(W)}^{\circ} \cap f(V)$. Our first move is $\beta'(\emptyset) = V_1$.

Now α' reacts to this by choosing $\emptyset \neq V_1' \subset V_1$, and we have to define $\beta'(V_1, V_1')$. We have $\overline{f(V_1')}^\circ \cap f(W) \neq \emptyset$. Hence we can find a nonempty open W_1 with $W_1 \subset W$ and $f(W_1) \subset f(W) \cap \overline{f(V_1')}^\circ$. The corresponding move is now $W_1 = \beta'(V_1, V_1')$.

Now player α' will react to this by providing $\emptyset \neq W_1' \subset W_1$. Since $f(W_1') \subset f(W_1) \subset \overline{f(V_1')}^\circ$, we have $\overline{f(W_1')}^\circ \cap f(V_1') \neq \emptyset$, so we pick a nonempty open V_2 with $V_2 \subset V_1'$ and $f(V_2) \subset f(V_1') \cap \overline{f(W_1')}^\circ$. Let $V_2 = \beta'(V_1, V_1', W_1, W_1')$ be our move.

Next player α' chooses $\emptyset \neq V_2' \subset V_2$, and we have to define $\beta'(V_1, V_1', W_1, W_1', V_2, V_2')$. From $f(V_2') \subset f(V_2) \subset \overline{f(W_1')}^\circ$ follows $\overline{f(V_2')}^\circ \cap f(W_1') \neq \emptyset$, so we can choose a nonempty open $W_2 \subset W_1'$ with $f(W_2) \subset \overline{f(V_2')}^\circ$. The move is now $\beta'(V_1, V_1', W_1, W_1', V_2, V_2') = W_2$. Etc.

Having defined β' in this way, let α' be a strategy which wins strongly against β'. Let $V_1 \supset V_1' \supset V_2 \supset V_2' \supset \dots$ and $W_1 \supset W_1' \supset W_2 \supset W_2' \supset \dots$ be the two nested sequences generated by the play. We have

i. $f(W_i') \subset f(W_i) \subset \overline{f(V_i')}^\circ$, $W_i' \subset W_i \subset W_{i-1}'$, $W_1 \subset W$;

ii. $f(V_{i+1}') \subset f(V_{i+1}) \subset \overline{f(W_i')}^\circ$, $V_i' \subset V_i \subset V_{i-1}'$, $V_1 \subset V$.

Now choose a sequence $w_i \in W_i'$. As α' is strongly winning against β', w_i has a cluster point $w \in \bigcap_{n=1}^\infty W_n \subset W$. Hence the sequence $f(w_i)$ has cluster point $f(w)$. Now for every open neighborhood G of $f(w)$ there exist numbers $n(G,1) < n(G,2) < \dots$ such that $f(w_{n(G,i)}) \in G$ for all i. By fact i. above $f(w_{n(G,i)}) \in \overline{f(V_{n(G,i)})}^\circ$, hence there exist $v_{n(G,i)} \in V_{n(G,i)}$ with $f(v_{n(G,i)}) \in G$ for all i. Let the set $\mathscr{G}$ of pairs (G,i) be ordered by $(G,i) \preceq (G',i')$ iff $G' \subseteq G$ and $i' \geq i$. Then as α' is strongly winning the net $\mathscr{N} = \langle v_{n(G,i)} : (G,i) \in \mathscr{G} \rangle$ has a cluster point $v \in \bigcap_{i=1}^\infty V_i \subset V$, hence $f(\mathscr{N})$ has cluster point $f(v)$. But by construction the net $f(\mathscr{N})$ converges to $f(w)$, hence $f(w) = f(v)$. As f is injective, we deduce $w = v$, and since $v \in V$, $w \in W$, $V \cap W \neq \emptyset$ follows. $\square$

Remark 3. Consider a space E possessing a web (T, ϕ) with the following property:

(μ) For every cofinal branch $b \subset T$, if $\bigcap_{t \in b} \phi(t) \neq \emptyset$ then every filter $\mathscr{F}$ containing all $\phi(t)$, $t \in b$, has an cluster point in $\bigcap_{t \in b} \phi(t)$.

Then in E the notions τ-Baire and τ^*-Baire coincide, and so do p- and c-completeness.

In particular, the notions τ-Baire and τ^*-Baire coincide in metrizable spaces, where (T, ϕ) with (μ) may be obtained by letting T the tree of finite sequences $(U_1, \dots, U_i)$ of non-empty open sets satisfying $\operatorname{diam}(U_i) \leq 1/i$ and $\overline{U}_i \subset U_{i-1}$, ordered by extension of sequences, with ϕ denoting $(U_1, \dots, U_i) \to U_i$.

7 Inclusion and quotient mapping theorem

In this section we ask how to extend the open mapping theorem to more general continuous nearly open mappings $f : E \to F$. We consider the cases (a) f injective and dense, i.e., $f(E)$ dense in F, but no longer surjective, (b) f surjective and no longer injective, and (c) f dense, and neither injective nor surjective. We start with case (a), where the answer is easy.

Proposition 3. *Let E be a B_r space and $f : E \to F$ a continuous, dense, injective and nearly open mapping into a Hausdorff space F. Then f is a homeomorphic embedding.*

Proof: It suffices to observe that f considered as a mapping $f : E \to f(E)$ is still nearly open, as follows from Lemma 3 below. Then by the definition of a B_r-space it is open, hence $E \simeq f(E)$. $\square$

Lemma 3. *Let $f : E \to F$ be a mapping such that $f(E)$ is dense in F. Then f is nearly open if and only if $f : E \to f(E)$ is nearly open.*

Proof: 1) Let f be nearly open as a mapping $E \to f(E)$. Let $V \subset E$ be open. Then the closure of $f(V)$ in $f(E)$ is $cl_{f(E)}f(V) = \overline{f(V)} \cap f(E)$. By assumption there exists O relatively open in $f(E)$ such that $f(V) \subset O \subset cl_{f(E)}f(V)$. Let $O = O_F \cap f(E)$ for O_F open in F. Then $f(V) \subset O_F \cap f(E) \subset cl_{f(E)}f(V) \subset \overline{f(V)}$. Hence $\overline{O_F \cap f(E)} \subset \overline{f(V)}$. But $f(E)$ is dense in F, hence $\overline{O_F \cap f(E)} = \overline{O}_F$, proving $f(V) \subset O_F \subset \overline{f(V)}$. Hence $f : E \to F$ is nearly open.

2) Conversely, suppose $f : E \to F$ is nearly open. Let $V \subset E$ be open. Then $f(V) \subset O_F \subset \overline{f(V)}$ for an open set O_F in F. Therefore $f(V) \subset O_F \cap f(E) \subset \overline{f(V)} \cap f(E)$. But $\overline{f(V)} \cap f(E) = cl_{f(E)}f(V)$, and $O = O_F \cap f(E)$ is relatively open in $f(E)$, hence $f : E \to f(E)$ is nearly open. $\square$

Concerning question (b) we have the following partial answer.

Theorem 5. *Let E be τ^*-Baire, $f : E \to F$ a continuous and nearly open surjection. Suppose f is factorized as $f = g \circ h$ with $h : E \to G$ a continuous surjection onto a semi-regular space G and $g : G \to F$ a continuous bijection. Then g is open.*

Proof: Note that g is nearly open, because if $U \subset G$ is open, then $h^{-1}(U)$ is open in E, hence $f(h^{-1}(U)) \subset \overline{f(h^{-1}(U))}^\circ$ by near openness of f. But clearly $f(h^{-1}(U)) = g(U)$.

Let $x \in G$ and U' a neighborhood of x. Using semi-regularity of G, choose an open neighborhood V' of x with $\overline{V'}^\circ \subset U'$. It suffices to prove $\overline{g(V')}^\circ \subset g(U')$, as this will show openness of g at x.

Let $y \in \overline{g(V')}^\circ$, $y = g(z)$. We will show $z \in \overline{V'}^\circ$, as this gives $z \in U'$. By continuity of g there exists an open O' with $z \in O'$ and $g(O') \subset \overline{g(V')}^\circ$. Proving $O' \subset \overline{V'}$ will now be sufficient. Take $w \in O'$ and an arbitrary open neighborhood W' of w, which may be assumed to satisfy $W' \subset O'$. It remains to prove $V' \cap W' \neq \emptyset$, as this will show $z \in \overline{V'}$.

Now let $U = h^{-1}(U')$, $V = h^{-1}(V')$, $W = h^{-1}(W')$. We define a strategy β' on E in exactly the same way as in the proof of Theorem 4. Following the proof all along will furnish elements $v \in \bigcap_{i=1}^\infty V_i \subset V$ and $w \in \bigcap_{i=1}^\infty W_i \subset W$ for which $f(v) = f(w)$. Now a difference occurs, as f is no longer injective. But g is, so from $f = g \circ h$ we obtain $h(v) = h(w)$. Hence $h(v) = h(w) \in h(V) \cap h(W) = V' \cap W'$, and that was to be shown. $\square$

For the following recall that the *semi-regularization* of a space E, denoted E_s, is the point-set E endowed with the coarser topology generated by the regular-open sets of E [8, p. 58]. The *regularization* of a space E, denoted E_r, is the point-set of E endowed with the finest regular topology coarser than the given one [33]. Note that E_r can be defined explicitly using the ultra-closure operator of [33], where the author calls E_r the associated regular space.

Suppose E is regular, $\sim$ an equivalence relation on E, $E/_\sim$ the quotient space, $\phi : E \to E/_\sim$ the quotient map. Since $E/_\sim$ is not necessarily regular, we use its regularization $(E/_\sim)_r$ and consider ϕ as a mapping $\phi : E \to (E/_\sim)_r$. We call ϕ a *regular-quotient* map and $(E/_\sim)_r$ the *regular-quotient*, because it preserves the following universal property of quotients: if $g : (E/_\sim)_r \to F$ is any mapping into a regular space F, then g is continuous iff $f = g \circ \phi$ is continuous. In the same vein, we call a continuous surjection f from a regular space E onto a regular space F *regular-quotient* if $F \simeq (E/_\sim)_r$ for the equivalence relation $x \sim y$ iff $f(x) = f(y)$.

Corollary 1. *Let E be a regular τ^*-Baire space, F a regular space, and $f : E \to F$ a continuous nearly open surjection. Then f is a regular-quotient mapping, i.e. the topology on F is the regularization, and also the semi-regularization, of the quotient topology.*

Proof: Consider the equivalence relation $x \sim y$ iff $f(x) = f(y)$ and let $E/_\sim$ be the quotient space with the usual quotient topology, $\phi : E \to E/_\sim$ the quotient map, $\widetilde{f} : E/_\sim \to F$ the continuous bijection satisfying $\widetilde{f} \circ \phi = f$. Then $E/_\sim$ is Hausdorff because F is, but $E/_\sim$ need not be semi-regular. Let $(E/_\sim)_s$ be the semi-regularization of $E/_\sim$. Then $(E/_\sim)_s$ is still Hausdorff and $\phi : E \to (E/_\sim)_s$ is continuous. The point is now that $\widetilde{f} : (E/_\sim)_s \to F$ remains continuous due to regularity of F, see [15, I.3.(3)] or [30, Prop. 2.2g]. We may therefore apply Theorem 5 with $G = (E/_\sim)_s$, $h = \phi$, $g = \widetilde{f}$, which shows that $\widetilde{f}$ is a homeomorphism. That gives $(E/_\sim)_s \simeq F$, so $(E/_\sim)_s$ is regular.

Now by definition the regularization of the quotient $(E/_\sim)_r$ carries the finest regular topology coarser than the quotient topology on $E/_\sim$, and since $(E/_\sim)_s$ was shown to be regular with a topology coarser than the quotient topology, we have continuity $(E/_\sim)_r \to (E/_\sim)_s \simeq F$. This means we can apply Theorem 5 again, now with $G = (E/_\sim)_r$, and now this implies $(E/_\sim)_r \simeq (E/_\sim)_s$, hence the claim $(E/_\sim)_r \simeq F$. $\square$

Remark 4. At first it seems that the correct setting for Corollary 1 ought to be semi-regular spaces, not regular spaces. Unfortunately, the semi-regularization E_s lacks the universal property of E_r, i.e., when $f : E \to F$ is continuous and $F = F_s$, then we do not necessarily get continuity of $f : E_s \to F$. What is amiss is that $\tau \subset \tau'$ on F does not imply $\tau_s \subset \tau'_s$. We could also say that the category of semi-regular Hausdorff spaces is not a reflective subcategory of the category of Hausdorff spaces.

Yet in the above situation we get $(E/_\sim)_s = (E/_\sim)_r$, hence $(E/_\sim)_s$ is quotient (extremal epimorphism) in the category of regular spaces, and at the same time has the convenient construction as semi-regularization of $E/_\sim$.

For completely regular spaces we proceed similarly. The complete-regularization of a given space E is the point-set E endowed with the finest completely regular topology coarser than the given one, denoted E_{cr}. Assuming that there exists at least one Hausdorff completely regular topology coarser than the given one, E_{cr} is Hausdorff. For E completely regular and $E/_\sim$ the usual quotient, we call $(E/_\sim)_{cr}$ a completely-regular quotient. Then a continuous surjection $f : E \to F$ onto a completely regular space F is completely-regular quotient if $F \simeq (E/_\sim)_{cr}$, where $x \sim y$ iff $f(x) = f(y)$.

Corollary 2. *Let E, F be completely regular and suppose E is a τ^*-Baire space. Let $f : E \to F$ be a continuous nearly open surjection. Then f is completely-regular quotient. Moreover, $F \simeq (E/_\sim)_{cr} = (E/_\sim)_s$.*

Remark 5. 1) The completely-regular quotient may also be characterized as the initial or limit topology with respect to $C(E/_\sim, \mathbb{R})$, i.e. the coarsest topology on $E/_\sim$ such that all real-valued functions continuous in the quotient topology are continuous.

2) Yet another way to describe completely-regular quotients is as follows. Let $\mathscr{U}, \mathscr{V}$ be uniformities on E, F inducing the topologies and such that f is uniformly continuous. (E.g. let $\mathscr{U}$ be the fine uniformity, then $\mathscr{V}$ does not matter). Then the topology on F is the one induced by the quotient uniformity [12], and this is $(E/_{\sim})_s$ under the assumptions of Corollary 2.

Application of Theorem 3 gives the following analogous:

Theorem 6. *Let E be a (completely) regular τ-Baire space, F a (completely) regular δ-space. Suppose $f : E \to F$ is a continuous nearly open surjection. Then f is a (completely) regular-quotient map.*

Remark 6. The reason why we cannot expect f to be open is that quotient maps in the category of Hausdorff spaces need not be open, and even when they are, $E/_{\sim}$ need not be semi-regular. One would need openness of $E \to (E/_{\sim})_s$ to deduce openness of f.

Our third question (c) has now an immediate answer. For $f : E \to F$ continuous nearly open and dense we can expect $f(E)$ with the topology induced from F to be a quotient of E, and this occurs under the hypotheses of Theorems 5 or 6.

The above findings motivate the following

Definition (Open mapping spaces)**.** Let $\mathscr{K}$ be a class of Hausdorff spaces. A Hausdorff space E is a $B(\mathscr{K})$-space, respectively, a $B_r(\mathscr{K})$-space, if every continuous nearly open surjection, respectively, bijection, f from E onto any $F \in \mathscr{K}$ is open.

A Hausdorff space E is a $B_q(\mathscr{K})$-space if every continuous nearly open surjection $f : E \to F$ onto any $F \in \mathscr{K}$ is a quotient map in the following sense: for any factorization $f = g \circ h$ with $h : E \to G$ continuous surjective onto a semi-regular Hausdorff space G and $g : G \to F$ continuous bijective, it follows that g is a homeomorphism. When $\mathscr{H}$ is the class of all Hausdorff spaces, then we say B_q-space instead of $B_q(\mathscr{H})$-space.

Remark 7. Every $B(\mathscr{K})$-space is a $B_q(\mathscr{K})$-space, and every semi-regular $B_q(\mathscr{K})$-space is a $B_r(\mathscr{K})$-space. Every B_r-space is semi-regular. For the latter, observe that the identity $i_E : E \to E_s$ is a continuous nearly open bijection. Since E_s is Hausdorff and E is B_r, i_E is open, so $E = E_s$.

Remark 8. Let $\mathscr{D}$ be the class of δ-spaces. Then Theorem 6 says that every semi-regular τ-Baire space is a $B_q(\mathscr{D})$-space, hence also a $B_r(\mathscr{D})$-space.

Example 7.1. Let $E = \mathbb{R} \setminus \mathbb{Q} \oplus \mathbb{R}$, $F = \mathbb{R}$, and let $f : E \to F$ be defined as $f|\mathbb{R} = i_{\mathbb{R}}$, $f|\mathbb{R} \setminus \mathbb{Q} = \iota_{\mathbb{R}\setminus\mathbb{Q}}$ the inclusion $\mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$. Then f is a continuous nearly open surjection which is not open, as the image of $\mathbb{R} \setminus \mathbb{Q}$ is not open in F. But E is completely metrizable, so f is a quotient map, which one can of course see directly as $E/_{\sim} \simeq \mathbb{R}$ for the equivalence relation $x \sim y$ iff $f(x) = f(y)$. So E is a B_q-space, hence a B_r-space, but not a B-space.

Remark 9. Recall that a Hausdorff space E is H-minimal [8, p. 223] if there is no strictly coarser Hausdorff topology on E, and it is H-closed if it is closed in every Hausdorff space containing E as a subspace. A space is H-minimal iff it is H-closed and semi-regular; [15, I.3]. Clearly H-minimal spaces are B_r-spaces.

Remark 10. In [11] the author constructs a first category H-minimal space which according to [25, p.593] is not a B-space. This is in contrast with the following:

Proposition 4. *Every H-minimal space is a B_q-space.*

Proof: Let $f = g \circ h$ with $h : E \to G$ surjective, $g : G \to F$ bijective, and G semi-regular. Since G is the continuous image of a H-closed space E, it is H-closed by a result of Katětov; cf. [8, p. 223], [15]. Since G is also semi-regular, it is H-minimal. In consequence g is a homeomorphism. $\square$

Theorem 7. *Let E be a semi-regular space containing a dense B_q-subspace D. Then E is a B_q-space.*

Proof: Let $f : E \to F$ be a continuous nearly open surjection, factorized as $f = g \circ h$ with $h : E \to G$ surjective, $g : G \to F$ bijective, and G semi-regular. We have to show that g is a homeomorphism.

Since D is dense in E, Lemma 4 below shows that the restriction $f|D$ to D remains nearly open as a mapping $D \to F$. But then since $f(D)$ is dense in F, Lemma 3 shows that $f|D$ is still nearly open as a mapping $D \to f(D)$.

Now consider the restriction $h|D : D \to h(D)$ of h on D, and the restriction $g|h(D) : h(D) \to f(D)$ of g on $h(D)$. Then $f|D = (g|h(D)) \circ (h|D)$ is a factorization with $g|h(D)$ bijective. Since $h(D)$ is dense in G, it is semi-regular, hence the factorization is amenable to Theorem 5. Since D is by hypothesis a B_q-space, $g|h(D)$ is a homeomorphism.

From here, based on near openness of g, denseness of $h(D)$ in G, and semi-regularity of G, we conclude using [25, Lemma p. 589] that g is also a homeomorphism. That completes the argument. $\square$

Lemma 4. *Let $f : E \to F$ be continuous and surjective. Let $V \subset E$ be open, $D \subset E$ dense. Then $\overline{f(V)} = \overline{f(V \cap D)}$.*

Proof: Let $y \in \overline{f(V)}$. Fix a neighborhood O_F of y in F. Then $O_F \cap f(V) \neq \emptyset$, hence $f^{-1}(O_F) \cap V$ is a nonempty open set in E. Since D is dense in E, $f^{-1}(O_F) \cap V \cap D \neq \emptyset$, and that implies $O_F \cap f(V \cap D) \neq \emptyset$. Since O_F was an arbitrary neighborhood of y, we have $y \in \overline{f(V \cap D)}$. $\square$

Remark 11. Theorem 7 marks a difference between B-spaces and B_q-spaces. For let E be a non-discrete Lindelöff P-space, then according to [25, §7] E is a B-space and so is the topological sum $E \oplus E$, while $E \oplus \beta E$ fails to be a B-space. Yet, due to the above, $E \oplus \beta E$ is a B_q-space.

Lemma 5. *Let $f = g \circ h$ with g, h both continuous and nearly open. Then f is nearly open.*

Proof: Let $x \in E$ and U a neighborhood of x. Since h is nearly open, we have $h(U) \subset O \subset \overline{h(U)}$ for some open O in G. Since g is also nearly open, we have $f(U) = g(h(U)) \subset g(O) \subset \overline{g(O)}^\circ \subset \overline{g(O)} \subset \overline{g(\overline{h(U)})} \subset \overline{g(h(U))} = \overline{f(U)}$ using continuity of g. $\square$

For this result see also [34, Lemma 8].

Proposition 5. *Let E be a B_q-space and h a continuous nearly open surjection onto a semi-regular space G. Then G is a B_r-space.*

Proof: Let $g : G \to F$ be a continuous nearly open bijection onto the space F. Then $f = g \circ h : E \to F$ is a continuous surjection, which by Lemma 5 is nearly open. As E is a B_q-space, g is a homeomorphism. That proves the claim. $\square$

8 Determined game

An infinite two-person game with two possible outcomes is called *determined* if either player α or player β has a winning strategy. The classical BM-game is not determined, as there exist Baire spaces which are not weakly α-favorable. In contrast, we shall see that for certain classes of spaces E the BM$'$-game is determined.

Proposition 6. *Let E be a metrizable locally convex vector space which is τ-Baire. Then E is complete.*

Proof: Since E is a B_r-space by Theorem 5, it is also a B_r-complete separated locally convex vector spaces in the sense of Pták. Hence, as the nomenclature suggests, E is complete, as follows from [17, §34, 2.(1)]. $\square$

This means the BM$'$-game is determined in the class of metrizable locally convex vector spaces E. Because if β' has no winning strategy, then E is τ-Baire, hence is completely metrizable by Proposition 6, so that by Proposition 1 player α' has a winning strategy. The same is true when we work in the category of separated topological vector spaces, where B- and B_r-completeness can be defined accordingly [1].

Proposition 7. *Let G be a metrizable topological group which is τ-Baire. Then G is completely metrizable.*

Proof: Let $\widetilde{G}$ be the completion of G in its two-sided uniformity. Suppose $\widetilde{x} \in \widetilde{G} \setminus G$. We form the space $F = G \cup \widetilde{x}G$ equipped with the topology induced from $\widetilde{G}$. Let $G \oplus G$ be the topological sum in the sense of [8, p. 74] and define the function $f : G \oplus G \to F$ as follows. Let the point-set of $G \oplus G$ be $G \times \{1\} \cup G \times \{2\}$, and let $f(x, 1) = x$, $f(x, 2) = \widetilde{x}x$. Then f is continuous and bijective, the latter since $G \cap \widetilde{x}G = \emptyset$. Since G and $\widetilde{x}G$ are both dense in $\widetilde{G}$, the mapping f is nearly open by Lemma 4. Since G is τ-Baire and metrizable, it is a B_r-space. By [26, Thm. 1] the topological sum $G \oplus G$ is also B_r, hence f is open, so that both G and $\widetilde{x}G$ are open in $\widetilde{G}$. That, however, is impossible. Hence $G = \widetilde{G}$. $\square$

Remark 12. The reasoning here differs from the one in Proposition 6, because B_r-groups need not be complete. For instance $Q = \{e^{2\pi i q} : q \in \mathbb{Q}\}$ is a H-minimal group, [10], hence a B_r-group. Clearly Q is not a B_r-space in the topological sense.

Remark 13. Proposition 7 shows that in a metrizable topological group, one of the players α', β' has a winning strategy, so again in that category the BM$'$-game is determined.

Remark 14. In Remark 2 we remarked that Baire spaces need not be τ-Baire. Now we remark that even spaces E where $E \times E$ is Baire need not be τ-Baire. Let E be a separable metrizable locally convex vector space which is Baire but not complete. Then $E \times E$ is Baire by a result of Kuratowski [8, p. 201], but by Proposition 6 above β' has a winning strategy, so E is not τ-Baire. This also shows that the converse of Proposition 2 is incorrect.

Remark 15. Kuratowski's result gives even E^n Baire for every n.

Let G be a topological group acting continuously on a Hausdorff space X. The G-flow on X is said to be minimal if every orbit Gx is dense in X. Then, expanding on the technique in Proposition 7, we have the following

Proposition 8. *Suppose the G-flow is minimal. Let $E = Gx$ be an orbit which is a B_r-space in the topology induced by X. Then Gx is not homeomorphic to any other orbit Gx'.*

Proof: Let $E = \{gx : g \in G\}$ for some $x \in X$, and suppose $E \simeq E' = \{gx' : g \in G\}$ for another $x' \in X$. Let $h : E \to E'$ be a homeomorphism. We let $E \oplus E$ be the topological sum of two copies of E represented on the set $E \times \{1\} \cup E \times \{2\}$. Then we define $f : E \oplus E \to X$ as $f(x, 1) = x$, $f(x, 2) = h(x)$. Then f is a continuous bijection onto the dense subset $E \cup E'$ of X, where injectivity is due to $E \cap E' = \emptyset$. Since $E \hookrightarrow X$ and $E' \hookrightarrow X$ are dense embeddings and h is a homeomorphism, f is nearly open. Since E is a B_r-space, so is $E \oplus E$. Hence f is a homeomorphism onto $E \cup E'$. But that means E, E' are both open in $E \cup E'$, contradicting the fact that they are both dense. $\square$

Remark 16. It is known [38] that if G is a Polish group, X a compact Hausdorff space, and if the G-flow on X is minimal and metrizable, then there exists an orbit Gx_0 which is residual in X. That means Gx_0 contains a dense G_δ subset of X, hence has a dense Čech-complete subspace, hence is a B_r-space [2, 25]. Then Gx_0 cannot be homeomorphic to any other orbit.

However, even when no residual orbit exists, as soon as we have a τ^*-Baire orbit, it is not homeomorphic to any other orbit.

9 Barely Baire spaces

We recall a construction from [9]. A subset $S \subset \omega_1$ is cofinal if $|S| = \omega_1$, and it is closed when closed in the order topology on ω_1. A subset $S \subset \omega_1$ is stationary if it intersects every closed cofinal subset of ω_1. Let ω_1 be endowed with the discrete topology and give ω_1^ω the product topology. Then ω_1^ω is metrizable. For $S \subset \omega_1$ cofinal put $S^* = \{f \in \omega_1^\omega : f^* := \sup_n f(n) \in S\}$, endowed with the topology induced by the product topology. Then by [9, Example 1, p. 234] S^* is a Baire space iff S is stationary. Moreover, if S, T are two stationary subsets of ω_1 such that $S \cap T$ is not stationary, then S^*, T^* are Baire, but $S^* \times T^*$ is not, hence S^*, T^* are what the authors of [9] call *barely Baire*. In addition, if we arrange $S \cap T = \emptyset$, then the topological sum $S^* \oplus T^*$ is no longer a B_r-space, nor is the product $S^* \times T^*$, see [9, p. 183]. In [9] it is also proved that S^* contains a dense completely metrizable subspace iff S is closed cofinal, which here due to metrizability is equivalent to weak α-favorability of S^*, or to p- and c-completeness.

Proposition 9. *Let S be stationary. Then S^* is a τ-Baire space, i.e., player β' has no winning strategy.*

Proof: Recall that sets of the form $B(\gamma) = \{f \in S^* : f(i) = \gamma_i, i = 1, \dots, k\}$ for finite sequences $\gamma = (\gamma_1, \dots, \gamma_k)$ of ordinals $\gamma_i < \omega_1$ form a basis for S^*. We consider a strategy β' playing with basic sets. Suppose $V = B(\gamma)$ and $V' = B(\gamma')$ are basic open sets with $V' \subset V$, then γ' is a prefix of γ. Therefore play sequences $V_1, V_1', W_1, W_1', \dots, V_r, V_r', \dots$

give rise to finite sequences of ordinals $\gamma_1 \subset \gamma_1' \subset \gamma_2 \subset \gamma_2' \subset \dots$ and $\delta_1 \subset \delta_1' \subset \delta_2 \subset \delta_2' \subset \dots$. We may therefore consider β' as a function on finite sequences of ordinals $\beta'(\gamma_1, \gamma_1', \delta_1, \delta_1', \dots, \delta_r, \delta_r') = \gamma_{r+1}$, respectively $\beta'(\gamma_1, \gamma_1', \delta_1, \delta_1', \dots, \gamma_r, \gamma_r') = \delta_r$.

We say that an ordinal $\eta < \omega_1$ is a fixed point of β' if $\max(\gamma_r) < \eta$, $\max(\delta_r) < \eta$ together with $\beta'(\gamma_1, \gamma_1', \delta_1, \delta_1', \dots, \delta_r, \delta_r') = \gamma_{r+1}$ imply $\max(\gamma_{r+1}) < \eta$, and at the same time $\max(\delta_{r-1}) < \eta$, $\max(\gamma_r) < \eta$ together with $\beta'(\gamma_1, \gamma_1', \delta_1, \delta_1', \dots, \gamma_r, \gamma_r') = \delta_r$, imply $\max(\delta_r) < \eta$.

We let F be the set of fixed points of β'. Clearly F is cofinal and closed. Now using stationarity of S, let $\eta \in S \cap F$. Fix a sequence $\eta_1 < \eta_2 < \dots$ converging to η. Now define $\alpha'(\gamma_1, \gamma_1', \delta_1, \delta_1', \dots, \gamma_r) = \gamma_r \frown \eta_r =: \gamma_r'$ and $\alpha'(\gamma_1, \gamma_1', \delta_1, \delta_1', \dots, \delta_r) = \delta_r \frown \eta_r =: \delta_r'$. Then the play of α' against β' generates two sequences $\delta_1 \subset \delta_1' \subset \delta_2 \subset \delta_2' \subset \dots$ and $\gamma_1 \subset \gamma_1' \subset \gamma_2 \subset \gamma_2' \subset \dots$ with $\sup_i \gamma_i = \sup_i \delta_i = \sup_i \eta_i = \eta$. Let $f = \bigcup_{i=1}^\infty \gamma_i$ and $g = \bigcup_{i=1}^\infty \delta_i$, then $f^* = g^* = \eta \in S$, hence $f, g \in S^*$. We have $f \in \bigcap_{k=1}^\infty B(\gamma_k)$ and $g \in \bigcap_{k=1}^\infty B(\delta_k)$, hence α' wins against β'. □

Corollary 3. *If S is stationary, then $S^* \times S^*$ is Baire, S^* is a B_q-space, and hence a B_r-space.*

Proof: The first statement is from [9]. The second statement follows with Proposition 9 in tandem with Theorem 6. □

Corollary 4. *Suppose S is stationary, but does not contain any closed cofinal set. Then neither α' nor β' have winning strategies, so in this class of spaces S^* the* BM′*-game is not determined.*

Proof: By Proposition 9 player β' has no winning strategy.

On the other hand, suppose α' has a winning strategy on S^*. Then by Proposition 1, so has α. As in the proof of Proposition 9 we can consider α a function on finite sequences of ordinals: $\alpha(\gamma_1, \delta_1, \gamma_2, \delta_2, \dots, \gamma_r) = \delta_r$, where $\gamma_1 \subset \delta_1 \subset \gamma_2 \subset \delta_2 \subset \dots$. Let F be the set of fixed-points of α, then F is closed cofinal in ω_1. We show that $F \subset S$. Indeed, let $\eta \in F$ and fix a sequence $\eta_1 < \eta_2 < \dots$ converging to η. Define a strategy β for the second player as follows: $\beta(\gamma_1, \delta_1, \dots, \gamma_r, \delta_r) = \delta_r \frown \eta_r =: \gamma_{r+1}$. Let $\gamma_1, \delta_1, \dots$ be the play of α against β. By the definition of F and the η_i we have $\sup_i \gamma_i = \sup_i \delta_i = \eta$. But α wins against β, so the function $f = \bigcup_i \gamma_i = \bigcup_i \delta_i$ belongs to S^*, which implies $\eta = f^* \in S$. That proves the second claim by contraposition. □

Remark 17. Amusingly, if we consider only cofinal sets $S \subset \omega_1$ which are Borel in the order topology of ω_1, then either S or $\omega_1 \setminus S$ contains a closed cofinal set. Hence in this class of spaces S^*, the BM′-game is again determined.

10 Barely τ-Baire spaces

We recall a second construction from [9]. Let $c = 2^\omega$, c^+ the successor cardinal of c, $C_\omega c^+$ the set of ordinals $\alpha < c^+$ with cofinality $\mathrm{cf}(\alpha) \leq \omega$. We select a family $\{A_y : y \in 3^\omega\}$ of mutually disjoint stationary subsets of $C_\omega c^+$. Then for $x \in 3^\omega$ define

$$B_x = \bigcup \{A_y : y \in 3^\omega, y(n) \neq x(n) \text{ for all } n \in \omega\}.$$

Note that this is in particular possible because $C_\omega c^+$ is itself a stationary subset of c^+. Now we endow 3 and c^+ with the discrete topology and define

$$E = \{(x, f) \in 3^\omega \times (c^+)^\omega : f^* = \sup_n f(n) \in B_x\},$$

endowed with the topology induced by the product topology. Since $3^\omega \times (c^+)^\omega$ is metrizable, so is E. Basic sets of E are indexed by finite sequences $\sigma = ((\alpha_0, \beta_0), \ldots, (\alpha_r, \beta_r)) \in (3 \times (c^+))^{r+1}$, that is,

$$B(\sigma) = \{(x, f) \in E : x(i) = \alpha_i, f(i) = \beta_i, i = 0, \ldots, r\}.$$

Following the line of [27, Example 3, p. 675] we have

Proposition 10. *The space E is τ-Baire.*

Proof: Let β' be a strategy in the BM$'$-game. We may assume that it plays with basic open sets $B(\sigma)$, $B(\tau)$. Again if $B(\sigma) \subset B(\sigma')$, then $\sigma' \subset \sigma$ is sequence extension. A play gives therefore rise to two sequences $\sigma_1 \subset \sigma_1' \subset \sigma_2 \subset \sigma_2' \subset \ldots$ and $\tau_1 \subset \tau_1' \subset \tau_2 \subset \tau_2' \subset \ldots$ intertwined as in the figure. Hence a strategy β' defines a mapping, also denoted β', with

$$\beta'(\sigma_1, \sigma_1', \tau_1, \tau_1', \ldots, \sigma_r, \sigma_r') = \tau_r, \quad \beta'(\sigma_1, \sigma_1', \ldots, \tau_r, \tau_r') = \sigma_{r+1}.$$

We can see this as two mappings Θ_1, Θ_2 on finite sequences $\sigma = (\sigma_1, \sigma_1', \sigma_2, \sigma_2', \ldots)$ and $\tau = (\tau_1, \tau_1', \tau_2, \tau_2', \ldots)$ such that $\Theta_1(\sigma, \tau) \supset \tau$ and $\Theta_2(\sigma, \tau) \supset \sigma$. Now let W be the set of ordinals $2 < \alpha < c^+$ with the following property:

There exist sequences (σ_n) and (τ_n) such that $\Theta_1(\sigma_0, \tau_0) = \sigma_1$, $\sigma_1 \subset \sigma_1'$, $|\sigma_1'| = |\sigma_1| + 1$, $\Theta_2(\sigma_1, \tau_0) = \tau_1$, $\tau_1 \subset \tau_1'$, $|\tau_1'| = |\tau_1| + 1$, etc., such that $\bigcup_n \sigma_n = (x, g)$, $\bigcup_n \tau_n = (y, h)$ with $g^* = h^* = \alpha$.

The set W is stationary in c^+. Indeed, if C is closed cofinal in c^+, define the extensions σ_i' of the σ_i and τ_i' of the τ_i such that $\max \sigma_i' \le \max \tau_{i+1}' \le \max \sigma_{i+1}'$ and such that $\sup_i \sigma_i' = \sup_i \tau_i' \in C$. Then $\bigcup_n \sigma_n = (x, g)$ and $\bigcup_n \tau_n = (y, h)$ have $g^* = h^* \in C \cap W$. This proves stationarity of W.

Now fix $z \in 3^\omega$ and let W_z be the set of those $\alpha \in W$ where (x, g), (y, h) exist as above, with $g^* = h^* = \alpha$, but $z(n) \notin \{x(n), y(n)\}$ for all n. Then

$$W = \bigcup \{W_z : z \in 3^\omega\}$$

and since W cannot be the union of $c < c^+$ stationary sets, one of the W_z is stationary. Now we use the following auxiliary result [9, Lemma 1]:

If $K \subset (c^+)^\omega$ is closed and $W = \{f^* : f \in K\}$ is stationary, then there exists a closed cofinal subset C of c^+ such that $C \cap C_\omega c^+ \subset W$.

We apply this to the stationary W_z found above in order to find $C \cap C_\omega c^+ \subset W_z$. For that let K be the set of all $((x, g), (y, h))$ with $\bigcup_n \sigma_n = (x, g)$ and $\bigcup_n \tau_n = (y, h)$, $g^* = h^* = \alpha$, for sequences as in the definition of W, having $z(n) \notin \{x(n), y(n)\}$ for every n. Then K is closed in $3^\omega \times (c^+)^\omega \times 3^\omega \times (c^+)^\omega$, and we have $W_z = \{\psi^* : \psi \in K\}$.

As the sets A_y arising in the construction of E are stationary, we have $W_z \cap A_z \cap C_\omega c^+ \neq \emptyset$. Choose γ herein and let (σ_n), (τ_n) be sequences as above giving rise to $\bigcup_n \sigma_n = (x, g)$, $\bigcup_n \tau_n = (y, h)$ with $g^* = h^* = \gamma$ and $z(n) \notin \{x(n), y(n)\}$. Since $\gamma \in A_z$, we have $(x, g), (y, h) \in E$. But (x, g) is in the intersection of the $B(\sigma)$, and (y, h) is in the intersection of the $B(\tau)$, hence both intersections are non-empty, and strategy α' is winning against strategy β'. That proves the claim. $\square$

Remark 18. The interest in this space is that $E \times E$ is Baire, (cf. Theorem 2), but $E \times E \times E$ is no longer Baire [9]. This gives a metrizable τ-Baire space E whose square $E \times E$ is no longer τ-Baire (even though $E \times E$ is Baire).

Secondly this gives rise to a metrizable B_r-space whose square is no longer B_r. Namely, either (a) $E \times E$ is not a B_r-space. Then E, which is B_r by Theorem 4, is the space we are looking for. Or (b) $E \times E$ is again B_r. Then $(E \times E) \times (E \times E)$ is no longer a B_r-space, because it is not Baire, but due to [27] ought to be Baire if it were B_r. So here we have the B_r-space $E \times E$ whose square is no longer B_r.

Remark 19. It would be interesting to know which of the two cases (a), (b) above is true. If (b) holds with $E \times E$ still B_r, then we have a metrizable B_r-space $E \times E$, which is Baire, but not τ-Baire. Currently no such space is known.

11 Michael game

The following variation of the Banach-Mazur game was introduced by Michael [18]. Players β and α choose successively non-empty sets $B_1 \supset A_1 \supset B_2 \supset A_2 \supset \dots$ such that A_i is open in B_i, that is, $A_i = B_i \cap U_i$ for some open set U_i in E. Player α playing with the sets A_i wins the game if $\bigcap_{n=1}^{\infty} \overline{A}_n \neq \emptyset$. Player α wins strongly if every filter $\mathscr{F}$ with $A_i \in \mathscr{F}$ for every i has a cluster point, i.e., $\bigcap\{\overline{F} : F \in \mathscr{F}\} \neq \emptyset$. In [18, Thm. 7.3] the author proves that α has a winning strategy if and only if E has a complete, exhaustive sieve.

We say that E is a m-Baire space if player β does not have a winning strategy in the Michael game. It is clear that every m-Baire space is Baire, because if player β plays with open sets, then α automatically responds with open sets, and the play coincides with the Banach-Mazur game.

Proposition 11. *Let E be a regular m-Baire space. Then every G_δ subset G of E is a m-Baire space.*

Proof: Let $G = \bigcap_{n=1}^{\infty} G_n$ with open sets $G_1 \supset G_2 \supset \dots$, and let β_G be a strategy for player β in the Michael game on G. We define a strategy β on the whole space.

Suppose $\beta_G(\emptyset) = B_1$. By regularity we may choose a set V_1 open in E such that $\overline{V}_1 \subset G_1$ and $B_1' = B_1 \cap V_1 \neq \emptyset$. Then define $\beta(\emptyset) = B_1'$. Now let $A_1 \subset B_1'$ be non-empty and relatively open in B_1', i.e., $A_1 = B_1' \cap U_1$ for some open U_1 in E. We have to define $\beta(B_1', A_1)$. Note that $A_1 \subset B_1' \subset B_1$ and $A_1 = B_1' \cap U_1 = B_1 \cap V_1 \cap U_1$, and since $A_1 \subset G$, $A_1 = B_1 \cap (V_1 \cap U_1 \cap G)$, so A_1 is relatively open in B_1 with regard to the space G. Hence $B_2 = \beta_G(B_1, A_1)$ is defined and is a non-empty subset of A_1. We choose a set V_2 open in E such that $\overline{V}_2 \subset G_2$ and $B_2' = B_2 \cap V_2 \neq \emptyset$. Define $\beta(B_1', A_1) = B_2'$. Etc.

By assumption E is m-Baire, so there exists a strategy α winning against β. Let us define a strategy α_G on G which wins against β_G. We have to define $\alpha_G(B_1)$, where $B_1 = \beta_G(\emptyset)$. We recall the construction of $B_1' \subset B_1$, $B_1' = \beta(\emptyset)$ and define $\alpha_G(B_1) = \alpha(B_1') = A_1$. Next we have to define $\alpha_G(B_1, A_1, B_2)$. We recall the construction of $B_2' \subset B_2$ and define $\alpha_G(B_1, A_1, B_2) = \alpha(B_1', A_1, B_2') = A_2$. Etc. It is clear from the construction that the play of α_G against β_G and the play of α against β are interlaced as follows: $B_1 \supset B_1' \supset A_1 \supset B_2' \supset B_2 \supset A_2 \supset \dots$. Since α wins, we have $\bigcap_{n=1}^{\infty} \overline{A}_n \neq \emptyset$, where the closure is with respect to E. Choose x within. Since $\overline{A}_i \subset \overline{B_i \cap V_i} \subset \overline{V}_i \subset G_i$, we have $x \in G$. But $A_i \subset G$, hence $cl_G(A_i) = \overline{A}_i \cap G$, hence $x \in cl_G(A_i)$. That shows

$\bigcap_{n=1}^{\infty} cl_G(A_n) \neq \emptyset$, hence α_G wins against β_G. □

Since in a metrizable space closed sets are G_δ-sets, we have the following

Corollary 5. *Every metrizable m-Baire space E is hereditary Baire. In particular, $E \times F$ is Baire for every Baire space F.*

Proof: This follows from a result of Moors [21]. □

Remark 20. As a consequence we see that the τ-Baire spaces S^* of Section 9 and [9] are not m-Baire, because if S, T are disjoint stationary subsets of ω_1, then S^*, T^* are metrizable Baire, but $S^* \times T^*$ is not Baire. Hence S^* is not hereditary Baire by [21], and so cannot be m-Baire.

In yet another well-known modification of the Banach-Mazur game player β chooses open sets V_k and points $x_k \in V_k$, while player α has to respond with open sets U_k satisfying $x_k \in U_k \subseteq V_k$. Player β wins when $\bigcap_{k=1}^{\infty} U_k = \emptyset$, otherwise α wins. The game is played with perfect information. The space is said to be strongly α-favorable if player α has a winning strategy [6]. We are interested in the undetermined case, i.e. when neither α nor β have winning strategies. This game was proposed by Choquet [3] and is thoroughly discussed in [6], and we write α^c, β^c for the corresponding strategies, all assumed to have complete memory. In the β^c-defavorable case we call the space c-Baire.

Proposition 12. *Every metrizable c-Baire space is m-Baire.*

Proof: Let E be metrizable and consider a strategy β^m in the Michael game. We define an associated strategy β^c. Suppose $\beta^m(\emptyset) = B_1$. We pick $b_1 \in B_1$ and choose an open set V_1 with $d(V_1) \leq 1/1$ and $b_1 \in V_1$. Put $\beta^c(\emptyset) = (b_1, V_1)$. Suppose now U_1 satisfies $b_1 \in U_1 \subset V_1$, so that it could be a move of α in response to the move $\beta^c(\emptyset)$. In that case we put $A_1 = B_1 \cap U_1$ and interpret A_1 as a move of α in response to $\beta^m(\emptyset)$.

Suppose now that $B_2 = \beta^m(B_1, A_1)$. If $A_1 = B_1 \cap U_1$ with U_1 constructed as above, then V_1 is available due to perfect memory, and we want to define $\beta^c((b_1, V_1), U_1) = (b_2, V_2)$. We pick $b_2 \in B_2$ and V_2 such that $b_2 \in V_2 \subset U_1$ satisfying $d(V_2) \leq 1/2$. Then (b_2, V_2) is our β^c move.

Continuing in this way gives a strategy β^c in the Choquet game. Let α^c be a strategy which wins against β^c. We use it to define a strategy α^m which wins against β^m. We have to define $\alpha^m(B_1, A_1, \dots, B_i)$, where $A_j = B_j \cap U_j$ for $1 \leq j \leq i-1$. In those cases where β^c has been derived from β^m as above, we have access to the corresponding (b_i, V_i), so we get $\alpha^c((b_1, V_1), U_1, (b_2, V_2), U_2, \dots, (b_i, V_i)) = U_i$. We then define $\alpha^m(B_1, A_1, \dots, B_i) = A_i := B_i \cap U_i$. In all other cases we define α^m at leisure.

Suppose now $B_1, A_1, B_2, A_2, \dots$ is the play of α^m against β^m. Then we get the play $(b_1, V_1), U_1, (b_2, V_2), U_2, \dots$ of α^c against β^c, and here α^c is winning, so we have $\bigcap_{i=1}^{\infty} U_i \neq \emptyset$. Pick $\bar{x}$ herein, then due to $d(V_i) \leq 1/i$ we have $b_i \to \bar{x}$, hence $\bar{x} \in \bigcap_{i=1}^{\infty} \overline{B}_i = \bigcap_{i=1}^{\infty} \overline{A}_i$, hence α^m wins against β^m. □

Remark 21. As a consequence, τ-Baire spaces S^* are not c-Baire either, i.e., for $S \subset \omega_1$ stationary but not containing a closed cofinal subset, S^* is β^c-favorable, even though a τ-Baire space.

Remark 22. A variant of the BM-game which bears some resemblance with our BM$'$-game is the Reznichenko-game of [22], but neither game seems stronger than the other.

12 Closed Graph Theorem

Moors [20, Thm. 2] proves that every nearly continuous closed graph mapping $f: E \to F$ from a Baire space E to a partition complete space F is continuous. Partition complete spaces, also known as cover complete spaces [18], are those where player α has a strong winning strategy in the Michael game.

The key observation here is that c-completeness of F is by a little margin too weak to prove the closed graph theorem, which is why in [28] a notion called strict c-completeness, equivalent to α-favorability in the strong Choquet game, was used. But that notion is now by a little margin too strong. The point made by [20] is that partition completeness, settled in between these twain, is just about right. This can also be seen in the light of Proposition 12 and Remark 21.

There are two ways to expand from here. We may introduce a tandem Michael game on F and weaken α-favorability to β'-defavorability, while strengthening Baire category of E to α-favorability. A second option is to keep the weaker hypotheses: Baire category of E and c-completeness of F, and require instead a little more on f. We shall follow this second line.

Lemma 6. *Let $f: E \to F$ be nearly open and nearly continuous with closed graph. Let E be Baire and F c-complete. Suppose G is a dense G_δ in F, V open in F, H a dense G_δ in E. Then $\overline{f^{-1}(V)} = \overline{f^{-1}(V \cap G) \cap H}$.*

Proof: It suffices to prove $f^{-1}(V) \subset \overline{f^{-1}(V \cap G) \cap H}$. Let $z \in f^{-1}(V)$, and take an open neighborhood W of z. We have to show $W \cap f^{-1}(V \cap G) \cap H \neq \emptyset$.

Let (T, ϕ) be a web on F satisfying (w_1), (w_2) and (c) from Section 2. Write $H = \bigcap_{n=1}^\infty H_n$ and $G = \bigcap_{n=1}^\infty G_n$ with G_n, H_n dense open and decreasing. We shall define a strategy β for the BM-game on E.

We have to define $\beta(\emptyset)$. The set $X_1 = \bigcup\{\phi(t) : t \in T, \phi(t) \subset V \cap G_1\}$ is dense in V, while $W \cap H_1$ is dense in W. Now $\overline{f(\overline{W \cap H_1})} \cap V$ is a neighborhood of $f(z)$, hence it intersects X_1, as X_1 is dense in V. By the definition of X_1 there exists x_1 in this intersection and $t_1 \in T$ with $x_1 \in \phi(t_1) \subset V \cap G_1$. Now $\phi(t_1) \cap f(\overline{W \cap H_1}) \neq \emptyset$. Choose $y_1 \in \overline{W \cap H_1}$ with $f(y_1) \in \phi(t_1)$. Since $\overline{f^{-1}(\phi(t_1))}$ is a neighborhood of y_1, it intersects $W \cap H_1$. Let z_1 be in this intersection. We pick an open set W_1 with $z_1 \in W_1 \subset W \cap H_1 \cap \overline{f^{-1}(\phi(t_1))}$. There exists $w_1 \in W_1 \cap f^{-1}(\phi(t_1))$. Let $\beta(\emptyset) = W_1$.

Let $W_1' \subset W_1$ be nonempty open, then we have to define $\beta(W_1, W_1')$. Now $D_2 = W_1' \cap H_2$ is dense in W_1', while $X_2 = \bigcup\{\phi(t) : t_1 <_T t, \phi(t) \subset G_2\}$ is dense in $\phi(t_1)$. Hence $\overline{f(\overline{D}_2)} \cap X_2$ is nonempty. By the definition of X_2 there exists x_2 in this intersection and t_2 with $t_1 <_T t_2$ and $x_2 \in \phi(t_2) \subset G_2$. That implies $\phi(t_2) \cap f(\overline{D}_2) \neq \emptyset$. Choose $y_2 \in \overline{D}_2$ with $f(y_2) \in \phi(t_2)$. Then $\overline{f^{-1}(\phi(t_2))}$ is a neighborhood of y_2, so cuts D_2. Choose $z_2 \in \overline{f^{-1}(\phi(t_2))} \cap D_2$ and an open W_2 with $z_2 \in W_2 \subset W_1' \cap H_2 \cap \overline{f^{-1}(\phi(t_2))}$. There exists $w_2 \in W_2 \cap f^{-1}(\phi(t_2))$. We let $\beta(W_1, W_1') = W_2$.

Continuing in this way defines a strategy β in the BM-game. Let α be a strategy winning against β. Let $W_1 \supset W_1' \supset W_2 \supset W_2' \supset \dots$ be their play. Then by construction we have

i. $D_i = W_{i-1}' \cap H_i$ dense in W_{i-1}',

ii. $X_i = \bigcup\{\phi(t) : t_{i-1} <_T t, \phi(t) \subset G_i\}$ dense in $\phi(t_{i-1})$, $x_i \in \phi(t_i) \subset G_i$, $t_{i-1} <_T t_i$.

iii. $y_i \in \overline{W_{i-1}' \cap H_i}$, $z_i \in W_i \subset W_{i-1}' \cap H_i \cap \overline{f^{-1}(\phi(t_i))}$, $w_i \in W_i \cap f^{-1}(\phi(t_i))$.

Since α is winning, there exists $w \in \bigcap_{i=1}^{\infty} W_i$. Let $\mathscr{N}$ be the set of pairs (N, k), where N is a neighborhood of w contained in W_k. For every such pair $N \cap f^{-1}(\phi(t_k)) \neq \emptyset$ by iii. We pick $w(N, k) \in N$ with $f(w(N, k)) \in \phi(t_k)$. Consider $\mathscr{N}$ directed by the relation $(N, n) \preceq (N', n')$ iff $N' \subseteq N$ and $n' \geq n$. Then the net $\mathcal{N} = \langle w(N, n) : (N, n) \in \mathscr{N} \rangle$ converges to w, while the net $f(\mathcal{N})$ converges to a point $v \in \bigcap_{i=1}^{\infty} \phi(t_i)$. Since the graph of f is closed, we deduce $v = f(w)$. But $f(w) \in \gamma(r_i) \subset G_i$ for every i implies $f(w) \in G$, while $f(w) \in \phi(t_1) \subset V$ gives $f(w) \in V$. On the other hand, $w \in W_k \subset H_k$ for every k gives $w \in H$. We have shown $w \in W \cap f^{-1}(V \cap G) \cap H$. $\square$

Theorem 8. *Let E be Baire, F regular and c-complete, $f : E \to F$ a nearly continuous and nearly open mapping with closed graph. Then f is continuous.*

Proof: By assumption F admits a web (T, ϕ) satisfying property (c) in the definition of c-completeness; cf. Section 2. Now let $x \in E$ and U a neighborhood of $f(x)$. By regularity of F choose an open neighborhood V of $f(x)$ with $f(x) \in V \subset \overline{V} \subset U$. Since $\overline{f^{-1}(V)}$ is a neighborhood of x, it suffices to prove $\overline{f^{-1}(V)} \subset f^{-1}(U)$. Let $y \in \overline{f^{-1}(V)}$, it suffices to prove $f(y) \in \overline{V}$. To this end let W be an open neighborhood of $f(y)$. It remains to prove $V \cap W \neq \emptyset$. We define a strategy β in the BM-game on E.

We have to define $\beta(\emptyset)$. We have $\overline{f^{-1}(V)} = \overline{f^{-1}(\bigcup\{\phi(t) : \phi(t) \subset V\})}$ by Lemma 6, and since $\overline{f^{-1}(W)}^\circ$ is a neighborhood of y, we have $\overline{f^{-1}(W)}^\circ \cap f^{-1}(\bigcup\{\phi(t) : \phi(t) \subset V\}) \neq \emptyset$. We choose $t_1 \in T$ and $z_1 \in \overline{f^{-1}(W)}^\circ$ with $f(z_1) \in \phi(t_1) \subset V$. Now $\overline{f^{-1}(W)} = \overline{f^{-1}(\bigcup\{\phi(s) : s \in T, \phi(s) \subset W\})}$ by Lemma 6, and since $\overline{f^{-1}(\phi(t_1))}^\circ$ is a neighborhood of z_1, it intersects $f^{-1}(\bigcup\{\phi(s) : s \in T, \phi(s) \subset W\})$. We choose $s_1 \in T$ with $\phi(s_1) \subset W$ and $y_1 \in \overline{f^{-1}(\phi(t_1))}^\circ$ such that $f(y_1) \in \phi(s_1) \subset W$. Then the open $U_1 = \overline{f^{-1}(\phi(t_1))}^\circ \cap \overline{f^{-1}(\phi(s_1))}^\circ$ is nonempty, and we let $\beta(\emptyset) = U_1$.

Now let $U_1' \subset U_1$ be nonempty open. We have to define $\beta(U_1, U_1')$. Since $U_1' \subset U_1 \subset \overline{f^{-1}(\phi(t_1))}^\circ$ and $\overline{f^{-1}(\phi(t_1))} = \overline{f^{-1}(\bigcup\{\phi(t) : t_1 <_T t\})}$ by Lemma 6, U_1' intersects $f^{-1}(\bigcup\{\phi(t) : t_1 <_T t\})$. Choose $t_2 \in T$ with $t_1 <_T t_2$ and $z_2 \in U_1' \cap f^{-1}(\phi(t_2))$. Now $f(z_2) \in \phi(t_2)$, hence $\overline{f^{-1}(\phi(t_2))}^\circ$ is a neighborhood of $z_2 \in U_1' \subset \overline{f^{-1}(\phi(s_1))}^\circ$. But $\overline{f^{-1}(\phi(s_1))} = \overline{f^{-1}(\bigcup\{\phi(s) : s_1 <_T s\})}$ by Lemma 6, hence $U_1' \cap \overline{f^{-1}(\phi(t_2))}^\circ$ intersects $f^{-1}(\bigcup\{\phi(s) : s_1 <_T s\})$. Find $s_2 \in T$ with $s_1 <_T s_2$ and $y_2 \in U_1' \cap \overline{f^{-1}(\phi(t_2))}^\circ$ such that $f(y_2) \in \phi(s_2)$. It follows that $U_2 = U_1' \cap \overline{f^{-1}(\phi(t_2))}^\circ \cap \overline{f^{-1}(\phi(s_2))}^\circ$ is nonempty, and we put $\beta(U_1, U_1') = U_2$.

Continuing to define β in this way, let α be a strategy winning against β. Then their play $U_1 \supset U_1' \supset U_2 \supset U_2' \supset \dots$ has the following properties:

1. $U_k' \subset U_k = U_{k-1}' \cap \overline{f^{-1}(\phi(t_k))}^\circ \cap \overline{f^{-1}(\phi(s_k))}^\circ$.
2. $z_{k+1} \in U_k' \cap f^{-1}(\phi(t_{k+1}))$, $t_k <_T t_{k+1}$.
3. $y_{k+1} \in U_k' \cap \overline{f^{-1}(\phi(t_{k+1}))}^\circ$, $f(y_{k+1}) \in \phi(s_{k+1})$, $s_k <_T s_{k+1}$.

Since α beats β, there exists $u \in \bigcap_{i=1}^{\infty} U_i \neq \emptyset$. Let $\mathscr{N}$ be the set of all pairs (N, k) where N is a neighborhood of u contained in U_k, ordered by the relation $(N, k) \preceq (N', k')$ iff $N' \subseteq N$ and $k' \geq k$. For $(N, k) \in \mathscr{N}$ we have $N \cap f^{-1}(\phi(t_k)) \neq \emptyset$, and also $N \cap f^{-1}(\phi(s_k)) \neq \emptyset$, due to property 1. above. Pick $v(N, k) \in N \cap f^{-1}(\phi(t_k))$ and $w(N, k) \in N \cap f^{-1}(\phi(s_k))$. Clearly the nets $\langle v(N, k) \rangle$ and $\langle w(N, k) \rangle$ both converge to u. On the other hand, by the completeness property (c) of the web (T, ϕ) the net $\langle f(v(N, k)) \rangle$ has a cluster point

$v \in \bigcap_{i=1}^{\infty} \phi(t_i) \subset V$ and similarly, the net $\langle f(w(N,k)) \rangle$ has a cluster $w \in \bigcap_{i=1}^{\infty} \phi(s_i) \subset W$, so $v \in V$ and $w \in W$. But by closedness of the graph we have $(u,v) \in \text{graph}(f)$ and $(u,w) \in \text{graph}(f)$, hence $v = w$, proving $V \cap W \neq \emptyset$. $\square$

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