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A Concrete Duality Approach to Compactness and Strict Singularity of Inclusion Operators

Dominikus Noll

1 Introduction

Let E, F be Banach sequence spaces satisfying $F \subset E$. We discuss properties of the inclusion operator $i : F \to E$ such as compactness, weak compactness, strict singularity and strict cosingularity. It is well-known that these properties admit dual descriptions in terms of the adjoint operator $i' : E' \to F'$. The theorems of Schauder and Gantmacher [4, p.485] reveal compactness and weak compactness as completely dual properties, while Pełczyński [18] shows that strict singularity and strict cosingularity are dually related in the sense that an operator is strictly singular (strictly cosingular) if its adjoint is strictly cosingular (strictly singular).

Dealing with sequence spaces, it is common to use, besides abstract topological duality, various notions of *concrete* sequence space duality, such as α- (or Köthe/Toeplitz),β- and γ-duality. Consequently, instead of expressing properties of $i : F \to E$ in terms of its adjoint i', one would then use the dual inclusion operators $i^\eta : E^\eta \to F^\eta$, where η stands for any one of the duality notions α, β, γ. In the present paper we pursue this idea, applying it to the above mentioned properties of $i : F \to E$.

Our approach is motivated by the fact that these operator theoretic properties are of interest also from the point of view of sequence space theory. Indeed, the importance of compact and weakly compact inclusions in summability theory was demonstrated among others by Bennett in [1], and by Schaffer/Snyder in [19]. Recently, in a series of papers [20,21,22], Snyder showed that strict singularity and strict cosingularity have sequence space interpretations. He discussed a sequence space property (noted $F < E$), which in many cases turned out to be a reformulation of strict cosingularity, and he indicated that, on the other hand, strict singularity of i is closely related with two sequence space properties known as the Meyer-König/Zeller property (noted MKZ) and the gliding humps property. Implicitely, both these properties were first discussed by Meyer-König and Zeller [9,10] in the context of summability theory. Their relation with strict cosingularity was further examined by the author in [12].

Using concrete duality instead of abstract topological duality has the advantage that the dual spaces E^η, F^η under consideration are again sequence spaces, a fact which in general is not true for the topological duals, and that the dual operators i^η are again inclusion operators, while i' is a restriction. A drawback of the concrete approach is that some information may be lost by using the smaller sequence space duals instead of the larger topological duals. Nevertheless, our approach is quite effective when $\beta-$ and γ-duality are used. It seems that $\alpha-$duality is somewhat too restrictive in the general context of sequence spaces. Its use was established by Köthe (cf. [8, vol.I]) in the frame of perfect sequence spaces.

In section 2 we start considering properties of γ-dual spaces needed later. Some of these results are of interest in themselves. In section 3 we study weakly compact inclusion operators, providing β- and γ-dual versions of Gantmacher's Theorem. As an application we obtain among others that a reflexive BK-space E has reflexive β-dual E^β if and only if it has sectional convergence. Section 4 investigates compact inclusion operators. We obtain β- and γ -dual versions of Schauder's Theorem.

An interesting aspect of concrete sequence space duality is given in section 5. We shed new light on the circle of problems connected with the Wilansky type properties recently discussed by G. Bennett, W. Stadler and the author. It turns out that, from the point of view presented here, these properties are β-dual versions of the Banach Homomorphism Theorem.

In section 6 we discuss strict singularity and strict cosingularity in the context of inclusion operators. We extend the results obtained by A.K. Snyder [20,21,22] and the author [12]. In the final paragraph 7 we present an example quoted as the Main Example, which turns out to be limiting for various results.

Our terminology is mainly based on the monograph [24]. Further references concerning notions from Functional Analysis and sequence space theory are [8,4,18,5,21,22,12]. The sections of a sequence $x \in \omega$ are denoted by $P_n x, n \in \mathbf{N}$.

2 γ -duals

This section is of a preparatory character. We state some results on β - and γ - dual spaces needed in the remainder of the paper. For basic facts concerning these duality notions we refer to [5], [24] and [6].

Recall that, given a BK-space E containing Φ, the β- dual space E^β is again a BK-

space under the norm

$$\| y \|_\beta = \sup_{\|x\|_E \leq 1} \sup_{n \in \mathbb{N}} \left| \sum_{i=1}^{n} x_i y_i \right|,$$

and similarly for $\| \cdot \|_\gamma$. Starting with $\| \cdot \|_\beta$ on E^β gives a norm $\| \cdot \|_{\beta\beta}$ on $E^{\beta\beta}$, and similarly for $E^{\gamma\gamma}$. In the following the spaces $E^\beta, E^\gamma, E^{\beta\beta}, E^{\gamma\gamma}$ will always be considered as BK-spaces with their corresponding norms defined as above. For the sake of clarity we will sometimes write $\| \quad \|_{E^\beta}$ instead of $\| \quad \|_\beta$.

The following result was essentially proved in [24, p.169]. We leave the necessary adjustments to the reader.

Lemma 2.1 *Let E be a BK-space containing Φ. Then the following four statements are equivalent:*

(1) E is closed in $E^{\beta\beta}$;

(2) E is closed in $E^{\gamma\gamma}$;

(3) E has an equivalent monotone norm (cf.[24, p.104]);

(4) Φ is a norming subspace of E'.

Recall that a linear subspace Y of the dual E' is called norming if $\|x\|_E = \sup\{|f(x)| : \|f\|_{E'} \leq 1, f \in Y\}$ is satisfied. Regarding Φ as a subspace of E' via natural identification, statement (4) above therefore means that we have $\|x\|_E = \sup\{|\langle x, y\rangle| : y \in \Phi, \|y\|_{E'} \leq 1\}$.

In contrast with β-dual spaces, the γ-dual spaces have an important feature already pointed out in [12]. Namely, they are always dual Banach spaces. Indeed, given a BK-space E containing Φ, let E_o denote the closure of Φ in $E^{\gamma\gamma}$. Then E_o is a BK-AK-space whose dual is $E_o' = E_o^\beta = E_o^\gamma = E^\gamma$ (cf.[24, 10.3.23]). This observation gives rise to the following

Lemma 2.2 *Let E, F be BK-spaces having $\Phi \subset F \subset E$. Then the following statements are equivalent:*

(1) E^β is closed in F^β;

(2) $E^\gamma = F^\gamma$;

(3) $E_o = F_o$.

Proof. Assume (1). First observe that E^γ is closed in F^γ. Indeed, as F^β is always closed in F^γ (cf. [24, p.158]), we deduce that E^β is closed in F^γ. Hence there exists $C > 0$ satisfying

$$\|x\|_{F^\gamma} \le \|x\|_{E^\beta} \le C \cdot \|x\|_{F^\gamma}$$

for all $x \in E^\beta$. Now for fixed $x \in E^\gamma$, this implies

$$\begin{aligned} \|x\|_{E^\gamma} &= \sup_n \|P_n x\|_{E^\gamma} \\ &= \sup_n \|P_n x\|_{E^\beta} \\ &\le C \cdot \sup_n \|P_n x\|_{F^\gamma} \\ &= C \cdot \|x\|_{F^\gamma}, \end{aligned}$$

proving that E^γ is closed in F^γ. Finally, observe that $E^\gamma = E_o^f, F^\gamma = F_o^f$, hence [24, 8.6.1] implies $E_o = F_o$.

Clearly (3) implies (2). Finally, (2) implies (1) since E^β is closed in E^γ, hence is closed in F^γ, and therefore is closed in F^β as well. □

We end this section with the following result, which gives some information on the interrelation between the space E and the corresponding space E_o.

Lemma 2.3 *Let E be a BK-space containing Φ, and let E_o be the closure of Φ in $E^{\gamma\gamma}$. Then*

(1) if E has AD, we have $E \subset E_o$;

(2) $E_o \subset E$ if and only if E has AB.

Proof. Concerning (1), note that the closure E_{AD} of Φ in E is always contained in the closure E_o of Φ in $E^{\gamma\gamma}$. As AD implies $E = E_{AD}$, the result follows.

Now consider statement (2). Observe that AB for E implies that E_{AD} has AK. Hence $E_{AD}^\gamma = E_{AD}^f = E^f$, the latter by [24, 7.2.4]. But AB also implies $E^f = E^\gamma$, hence $E_o^\gamma = E^\gamma = E_{AD}^\gamma$. Finally, [24, 8.6.1] implies $E_o = E_{AD}$, giving $E_o \subset E$.

Conversely, assume $E_o \subset E$. This implies $E_o \subset E_{AD}$, hence $E_o = E_{AD}$. Therefore $E^f = E_{AD}^f = E_o^f = E_o^\gamma = E^\gamma$, proving that E has AB. □

Remarks. 1) As a consequence of statement (2) of Lemma 2.3 we find that the class of all BK-AB-spaces F having γ-dual $F^\gamma = E^\gamma$, has a largest element, $E^{\gamma\gamma}$, and a smallest element, E_o. In particular, this implies $F_o = E_o$ for all BK-AB-spaces F in this class. Notice that $E^{\beta\beta}$ also has $\gamma-$ dual E^γ (cf.[5]), so $E_o \subset E^{\beta\beta} \subset E^{\gamma\gamma}$. This proves that E_o could as well have been defined as the closure of Φ in $E^{\beta\beta}$. However, E_o^β in general is different from E^β, so it is not clear whether a smallest BK-AB-space with β-dual E^β exists in general.

2) The converse of statement (1) of Lemma 2.3 is not valid. Indeed, let E be a proper dense BK-subspace of ℓ_1 having $E^\beta = \ell_1^\beta = \ell_\infty$. A space of this type may be

$E = \{x \in \ell_1 : ((2n)^2 x_{2n} - (2n+1)^2 x_{2n+1}) \in c_o\}$ (see also [2, Theorem 6]). Then we have $E_o = \ell_1$, but E does *not* have AD, for the latter would imply $E = \ell_1$. For details concerning the space E above we refer to the main example in section 7.

3 Weak Compactness

The purpose of this section is to prove β- and γ-dual versions of Gantmacher's Theorem, which states that an operator T between Banach spaces is weakly compact if and only if its adjoint T' is. Here, one difficulty consists in finding the right substitutes for the weak topologies $\sigma(E', E'')$ and $\sigma(F', F'')$. Before presenting the solution, we need the following

Lemma 3.1 *Let F be a BK-space containing Φ. Then the following statements are equivalent:*

(1) F has AB;

(2) The $\sigma(F^{\beta\beta}, F^{\beta})$-closure $B^{\bullet\bullet}$ of the unit ball B of F is absorbing in $F^{\beta\beta}$. (Here $^\bullet$ stands for the polar with respect to the dual pairing $\langle F^{\beta\beta}, F^{\beta}\rangle$).

Proof. Observe that F has AB if and only if F^{β} is closed in F', i.e. when the dual norm and the β-dual norm are equivalent on F^{β}. Now, as the polar $B^\bullet$ of the unit ball B of F with respect to the pairing $\langle F^{\beta\beta}, F^{\beta}\rangle$ is just the dual unit ball intersected with F^{β}, i.e.

$$B^\bullet = B^\circ \cap F^\beta,$$

(where $^\circ$ stands for the polar in the dual pairing $\langle F, F'\rangle$), we deduce that AB is equivalent with the statement

$$B^\bullet \subset \rho \cdot D \qquad \text{for some} \qquad \rho > 0$$

where $D = \{y \in F^\beta : \|y\|_\beta \leq 1\}$. This gives

$$D^\bullet \subset \rho \cdot B^{\bullet\bullet}.$$

As $D^\bullet$ is the $\|\cdot\|_{\beta\beta}$-unit ball in $F^{\beta\beta}$, hence is absorbing, the equivalency of (1) and (2) follows. □

Theorem 3.2 *Let E, F be BK-spaces having $\Phi \subset F \subset E$. Suppose F has AB and $i : F \to E$ is weakly compact. Then $i^{\beta\beta} : F^{\beta\beta} \to E^{\beta\beta}$ maps $F^{\beta\beta}$ into E and is weakly compact as a mapping $F^{\beta\beta} \to E$.*

Proof. Let B be the unit ball in F, and let C denote the closure of $i(B)$ in E. Then C is $\sigma(E,E')$-compact. Hence C is also $\sigma(E,E^\beta)$-compact, for this topology is weaker. Consequently, C is the closure of $i(B)$ with respect to the topology $\sigma(E^{\beta\beta},E^\beta)$. Now by Lemma 3.1 above the $\sigma(F^{\beta\beta},F^\beta)$-closure $\overline{B}=B^{\bullet\bullet}$ of B in $F^{\beta\beta}$ is absorbing, hence is a neighbourhood of 0 in the Banach space $F^{\beta\beta}$. On the other hand, the mapping $i^{\beta\beta}$ is continuous with respect to the topologies $\sigma(F^{\beta\beta},F^\beta)$ and $\sigma(E^{\beta\beta},E^\beta)$, so we have

$$i^{\beta\beta}(B^{\bullet\bullet}) = i^{\beta\beta}(\overline{B}) \subset \overline{i^{\beta\beta}(B)} = \overline{i(B)} = C,$$

where we use $i^{\beta\beta} \restriction F = i$, and where the bar refers to the $\sigma(F^{\beta\beta},F^\beta)$ and $\sigma(E^{\beta\beta},E^\beta)$-closures respectively. This proves that $i^{\beta\beta}(B^{\bullet\bullet})$ is weakly relatively compact in E, and that $i^{\beta\beta}(F^{\beta\beta}) \subset E$. This ends the proof. □

We shall now derive a β-dual version of Gantmacher's theorem which relates weak compactness of i to weak compactness of i^β.

Theorem 3.3 *Let E,F be BK-spaces having $\Phi \subset F \subset E$. Suppose E has AB and $i^\beta : E^\beta \to F^\beta$ is weakly compact. Then so is $i : F \to E$.*

Proof. a) Let B be a neighbourhood of 0 in E^β. As E has AB, E^β is closed in E', so B may be chosen of the form

$$B = B' \cap E^\beta,$$

where B' denotes the dual unit ball in E'. Then the image of B under i^β is relatively compact in the weak topology $\sigma(F^\beta, F^{\beta\prime})$. Let C denote the closure of $i^\beta(B)$ in this topology. We claim that C is as well compact with respect to the weak topology $\sigma(F',F'')$. Indeed, as $F^\beta \to F'$ is a continuous inclusion, it is also continuous with respect to the corresponding weak topologies, hence the image of C under this inclusion is weakly compact in F'. This proves the claim.

b) Observe that as a consequence of a) above, C is also compact with respect to the weak star topology $\sigma(F',F)$, in particular, C is the closure of $i^\beta(B)$ with respect to $\sigma(F',F)$.

c) Denoting the $\langle E',E\rangle$-polar by $^\circ$, the set $B^{\circ\circ}$ is the closure $\overline{B}$ of B in E' with respect to the weak star topology $\sigma(E',E)$. But notice that the restriction $i' : E' \to F'$ is weak star continuous, hence we obtain

$$i'(B^{\circ\circ}) = i'(\overline{B}) \subset \overline{i'(B)} = \overline{i^\beta(B)} = C,$$

where the last equality follows from the fact that C is also the weak star closure of $i^\beta(B)$, see part b), and where we use $i' \restriction E^\beta = i^\beta$. The proof will consequently be complete if we prove that $B^{\circ\circ}$ is a neighbourhood of 0 in E', for this shows the weak compactness of i', and hence the weak compactness of i by Gantmacher's theorem.

d) Notice that $B^{\circ} = B'^{\circ}$ by the choice of B and the fact that E^{β} is weak star dense in E'. Hence $B^{\circ\circ} = B'^{\circ\circ} = B'$. This completes our argument. □

Corollary 3.4 *Let E, F be BK-spaces having $\Phi \subset F \subset E$. Suppose F has AB and $i : F \to E$ is weakly compact. Then so is $i^{\beta} : E^{\beta} \to F^{\beta}$.*

Proof. Theorem 3.2 implies that $i^{\beta\beta} : F^{\beta\beta} \to E^{\beta\beta}$ is weakly compact. Hence by Theorem 3.3, $i^{\beta} : E^{\beta} \to F^{\beta}$ is weakly compact. □

The following is a nice consequence which could be expected from Gantmacher's Theorem, namely from the version stating that $T : F \to E$ is weakly compact if and only if $T''(F'') \subset E$.

Corollary 3.5 *Let E be a reflexive BK-AB-space. Then $E^{\beta\beta} = E$. In particular, E has a monotone norm.*

Proof. Apply Theorem 3.2 to the inclusion $i : E \to E$, which is weakly compact here. □

Remarks. 1) The AB assumption on the space F in Theorem 3.2 respectively the space E in Corollary 3.5 may not be omitted. Indeed, in the case of Corollary 3.5 this is immediately clear from the fact $E = E^{\beta\beta}$ implies that E has AB, while in the case of Theorem 3.2 this is best seen by considering again the main example (section 7).

2) The condition $F^{\beta\beta} \subset E$ in Theorem 3.2 is clearly not sufficient to imply weak compactness of the inclusion $F \to E$, so the naiv analogue of Gantmacher's result fails in this situation. Take for instance $F = \ell_1, E = c_o$, then $F = F^{\beta\beta} \subset E$, but $\ell_1 \to c_o$ is not weakly compact.

One might conjecture that weak compactness of an inclusion $F \to E$ implies weak compactness of $E^{\beta} \to F^{\beta}$ in the case where F is an AD-space. Actually, the compact version of this statement is valid, as we will prove in the next section (Theorem 4.1). But the weakly compact version fails, as we will see below. First we have the following

Lemma 3.6 *Let E be a BK-space containing Φ. Suppose E^{β} is reflexive. Then $E = E_o$, and E is reflexive.*

Proof. Reflexivity of E^{β} implies weak compactness of $E^{\beta} \to E^{\beta}$. Hence, by Theorem 3.3, $E^{\beta\beta} \to E^{\beta\beta}$ is weakly compact, so $E^{\beta\beta}$ is reflexive. As E_o is the closure of Φ in $E^{\beta\beta}$ (cf. section 2), E_o is reflexive, and hence so is its dual $E'_o = E^{\gamma}_o = E^{\gamma}$. This implies $E^{\gamma\prime} = E_o$. Consequently, we have $E^{\gamma\gamma} = E_o$. Indeed, given any $y \in E^{\gamma\gamma}$, there exists

$f \in E^{\gamma'}$ satisfying $f(x) = \langle x, y \rangle$ for all $x \in \Phi$. But $E^{\gamma'} = E_o$ implies the existence of a $z \in E_o$ having $f(x) = \langle x, z \rangle$ for all $x \in E^{\gamma}$. This gives $y = z \in E_o$.

So $E_o = E^{\gamma\gamma}$. But recall that we always have $E_o \subset E^{\beta\beta} \subset E^{\gamma\gamma}$. This implies $E_o = E^{\beta\beta}$, hence $E_o^{\beta} = E^{\beta}$. In particular, this implies $E \subset E_o$, for $E \subset E^{\beta\beta}$ is always true.

Now observe that E_o is a BK-AK-space whose β-dual is E^{β}. The latter space is itself AK, since $E^{\beta\prime} = E^{\beta\beta}$. Hence the space E_o has the Wilansky property (see [2,23] and section 5). This means that every dense BK-subspace of E_o having the same β-dual must coincide with E_o. But notice that E is of this type, so $E = E_o$. This gives the result. □

The proof of Lemma 3.6 contains more information than the statement. In fact, we obtain the following result improving Corollary 3.5.

Theorem 3.7 *Let E be a BK-space containing Φ. Then the following statements are equivalent: (1) E^{γ} is reflexive, (2) E^{β} is reflexive, (3) $E^{\gamma\gamma}$ is reflexive, (4) $E^{\beta\beta}$ is reflexive, (5) E_o is reflexive, (6) E is reflexive and has AB, (7) E is reflexive and has AK.*

Remarks. 1) Let E be a reflexive BK-AD-space which does not have AK. Then $E \to E$ is weakly compact, but $E^{\beta} \to E^{\beta}$ and $E^{\gamma} \to E^{\gamma}$ are not by Theorem 3.7. This provides an example for the fact that weak compactness if i does *not* imply weak compactness of i^{β}, i^{γ} even when the source space has AD. A reflexive BK-AD-space E not having AK may be obtained by taking the domain of ℓ_2 with respect to the Cesàro matrix C_2 of second order.

2) We give an example indicating that Theorem 3.7 is not valid for α-duality. Indeed, there exists a non-reflexive BK-AD-space E contained in ℓ_2 and satisfying $E^{\alpha} = \ell_2^{\alpha} = \ell_2$. Let F be the domain of the BK-space $\ell_2 + line$ with respect to the summation matrix S, then F is a reflexive BK-AD-space with $\Phi \subset F \subset \ell_2$ satisfying $F^{\alpha} = \ell_2^{\alpha} = \ell_2$. Clearly F is a proper dense subspace of ℓ_2. Now let E be any non-reflexive proper dense BK-subspace of ℓ_2 containing F. Then E is as desired.

3) The results of this section remain valid if β-duality is replaced by γ-duality. In the case of Theorem 3.3 this is immediately clear, since weak compactness of i^{γ} implies weak compactness of i^{β}, for F^{β} is closed in F^{γ}. Also the γ-dual version of Theorem 3.2 follows by making appropriate changes in the proof.

4 Compactness

The classical Theorem of Schauder [4, p.485] states that an operator T between Banach spaces is compact if and only if its adjoint T' is. Here we obtain analogues of Schauder's result for β- and γ-duality.

Theorem 4.1 *Let E, F be BK-spaces satisfying $\Phi \subset F \subset E$. Suppose F has AD and the inclusion $i : F \to E$ is compact. Then so is $i^\beta : E^\beta \to F^\beta$.*

Proof. a) Let $\epsilon > 0$ be fixed. Using the fact that i is compact and Φ is dense in F, we find vectors $y^1, \ldots, y^n \in \Phi$ having $\|y^i\|_F \leq 1$ such that, given any $y \in F, \|y\|_F \leq 1$, there exists $i \in \{1, \ldots, n\}$ such that $\|y - y^i\|_E \leq \epsilon/3$.

b) As the vectors y^i have been chosen from Φ, there are only finitely many different vectors $P_r y^i$. Let us arrange these as a finite sequence $z^1, \ldots, z^s$. Now consider the linear operator $x \to (\langle x, z^1\rangle, \ldots, \langle x, z^s\rangle), E^\beta \to \mathbf{C}^s$. As the range space is finite dimensional, the operator is compact, so there exist vectors $x^1, \ldots, x^m \in E^\beta, \|x^j\|_{E^\beta} \leq 1$ such that given any $x \in E^\beta, \|x\|_{E^\beta} \leq 1$, we find $j \in \{1, \ldots, m\}$ satisfying

$$\sup_{i=1,\ldots,s} |\langle z^i, x - x^j\rangle| \leq \epsilon/3.$$

Hence by the definition of the z^i we have

$$\sup_{r\in\mathbf{N}} \sup_{i=1,\ldots,n} |\langle P_r y^i, x - x^j\rangle| \leq \epsilon/3.$$

c) Now let $x \in E^\beta, \|x\|_{E^\beta} \leq 1$ be fixed. Choose $j \in \{1, \ldots, m\}$ for $\epsilon/3$ and x as in b). Now fix $y \in F, \|y\|_F \leq 1$. Choose $i \in \{1, \ldots, n\}$ according to a). Then we have

$$\begin{aligned} |\langle P_r y, x - x^j\rangle| &\leq |\langle y - y^i, P_r x\rangle| + |\langle y^i, P_r(x - x^j)\rangle| + |\langle y - y^i, P_r x^j\rangle| \\ &\leq \|y - y^i\|_E \|P_r x\|_{E^\beta} + \epsilon/3 + \|y - y^i\|_E \|P_r x^j\|_{E^\beta} \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3. \end{aligned}$$

This implies

$$\|x - x^j\|_{F^\beta} = \sup_{\|y\|_F \leq 1} \sup_{r\in\mathbf{N}} |\langle P_r y, x - x^j\rangle| \leq \epsilon,$$

hence i^β is compact. □

Remark. The AD assumption on F may not be omitted here. This may be seen from the following example. Let $F = c_o + \text{lin}\{a\}$, where a is the sequence having $a_n = n^2$, and let $E = \{x \in \omega : (x_n/n^2) \in bv\}$, both given their natural BK-topologies. Clearly $\Phi \subset F \subset E$. As $F \cong c_o$ and $E \cong \ell_1$, the inclusion $F \to E$ is compact (cf.[8, §42,3.(9).]). But notice that $E^\beta = \{y \in \omega : (n^2 x_n) \in cs\} = F^\beta$, so the inclusion i^β is the identity operator, hence is certainly not compact.

Corollary 4.2 *Let E, F be BK-spaces satisfying $\Phi \subset F \subset E$. Suppose F has AB and the inclusion $i : F \to E$ is compact. Then so is $i^\beta : E^\beta \to F^\beta$.*

Proof. Let F_{AD} be the closure of Φ in F, then F_{AD} has AK and the inclusion $F_{AD} \to E$ is compact. Hence by Theorem 4.1 above, $E^\beta \to F^\beta_{AD}$ is compact. But notice that $F^\beta_{AD} = F^f$, and that $F^f = F^\gamma$ as F has AB. So the inclusion $E^\beta \to F^\gamma$ is compact. As F^β is closed in F^γ, the result follows. □

Corollary 4.3 *Let E, F be $BK-$spaces having $\Phi \subset F \subset E$. Suppose E has a monotone norm. Let $i^\beta : E^\beta \to F^\beta$ be compact. Then $i : F \to E$ is compact.*

Proof. Since E^β has AB, it follows from Corollary 4.2 above that $F^{\beta\beta} \to E^{\beta\beta}$ is compact, hence the inclusion $F \to E^{\beta\beta}$ is compact. Finally, the fact that E has monotone norm implies that E is closed in $E^{\beta\beta}$ (see Lemma 2.1), and this proves the compactness of the inclusion $F \to E$. □

Remarks. 1) Theorem 4.1 and its Corollaries remain valid if β-duality is replaced by γ-duality. This is immediate for Corollary 4.3, since the β-duals are closed in the corresponding γ-duals, so compactness of $E^\gamma \to F^\gamma$ readily implies compactness of $E^\beta \to F^\beta$. Concerning Theorem 4.1, observe that the proof of the γ-dual version is essentially the same. Finally, the γ-dual version of Corollary 4.2 results from the fact that AB for the space F implies $F^\gamma = F^f$.

2) We consider the following example. Let $F = \{x \in \omega : (n^2 x_n) \in c_o\}, E = \{x \in \ell_1 : ((2n)^2 x_{2n} - (2n+1)^2 x_{2n+1}) \in c_o\}$. Then the inclusion $F \to E$ is not compact. Nevertheless, $E^\beta = \ell_\infty, F^\beta = \{y : (y_n/n^2) \in \ell_1\} \cong \ell_1$, so the inclusion $E^\beta \to F^\beta$ is compact. This proves that Corollary 4.3 is not even valid in the case where F is BK-AK, i.e., some restictive requirement on the space E is needed. For details concerning this example we refer to section 7.

We end this section with the following result improving Corollary 4.3.

Proposition 4.4 *Let E, F be BK-spaces having $\Phi \subset F \subset E$. Suppose E has AB and $i^\beta : E^\beta \to F^\beta$ is compact. Then so is $i : F \to E$.*

Proof. The method of proof is essentially the same as in Theorem 3.3. Using a similar argument, here one proves compactness of i' and then uses Schauder's theorem. □

5 Homomorphism Theorem

The homomorphism theorem for Banach spaces states that an operator $T : F \to E$ is a homomorphism, i.e. an open mapping onto $T(F)$, if and only if its adjoint T' is a homomorphism (cf.[8, §33,4.(1).]), or equivalently, T has closed range if and only if T' has closed range.

In a series of papers, G. Bennett [2], W. Stadler and the author [23,14,15,16,17,11,12] have studied what turns out to be β- and γ-dual versions of the homomorphism theorem. The situation may be described more precisely by the following definition given by Bennett [2].

A BK-space E containing Φ is said to have the *Wilansky property (W1)* if every BK-subspace F of E satisfying $F^\beta = E^\beta$ is closed in E. E is said to have the *Wilansky property (W)* if every dense BK-subspace F of E satisfying $F^\beta = E^\beta$ coincides with E. Replacing β-duality by γ-duality here leads to the Wilansky properties $(\gamma - W1)$ and $(\gamma - W)$ respectively. It was proved in [13] that the Wilansky properties (W) and $(\gamma - W)$ are equivalent.

Observe that in view of Lemma 2.2, the Wilansky property $(\gamma - W1)$ is just one half of the β- and γ-dual versions of the homomorphism theorem for inclusions $i : F \to E$, while the Wilansky property (W) may be regarded as the corresponding part of the homomorphism theorem for dense inclusions. Notice that the Wilansky properties above are not valid for all BK-spaces (see for instance [2]), while the homomorphism theorem clearly *is* a general statement. This shows again that we have to be careful about taking for granted the validity of β- and γ-dual versions of results familiar in Functional Analysis.

The results quoted above deduce the homomorphism property of i from the fact that i^β is a homomorphism. In the following we provide a result of the reverse type deriving the homomorphism property of i^β from the fact that i is a homomorphism.

Proposition 5.1 *Let E be a BK-AB-space. Let F be a closed subspace of E containing Φ. Then E^β is closed in F^β. In other terms, if $i : F \to E$ is a homomorphism, then so is $i^\beta : E^\beta \to F^\beta$.*

Proof. As F is closed in E, we have $F^f = E^f$, hence $F^\gamma = E^\gamma$, since E, F have AB by assumption. But now Lemma 2.2 gives the result. □

The statement of the Proposition is no longer true if the AB assumption on E is omitted. This may be seen from the following example. Let $F = c_o$, $E = c_o + \mathrm{lin}\{a\}$, where $a_n = n^2$, then we have $E^\beta = \{x : (n^2 x_n) \in cs\}$, so $i : F \to E$ is a homomorphism, but $i^\beta : E^\beta \to F^\beta = \ell_1$ is not.

6 Strict Singularity and Cosingularity

An operator between Banach spaces $T : F \to E$ is called ***strictly singular*** if for no infinite dimensional subspace S of F the operator $T \restriction S : S \to T(S)$ is an isomorphism. This notion has been introduced by Kato [7]. Dually, the operator T is called ***strictly cosingular*** if for no infinite codimensional closed subspace M of E the operator $q \circ T : F \to E/M$

is an epimorphism (i.e. is surjective), where q denotes the quotient mapping $E \to E/M$. This notion has been introduced by Pełczyński [18]. We refer to [3] for a survey of results and references concerning both these notions.

Topological duality of strict singularity and cosingularity is not complete, as it is in the case of weak compactness and compactness. It is known (cf.[18]) that strict singularity (strict cosingularity) of the adjoint T' implies strict cosingularity (strict singularity) of T. Implications of the reverse type need additional assumptions on the spaces E or F. Recently, Snyder [22], using techniques from sequence space theory, has obtained a complete duality result for strict cosingularity in the case where the space F is separable and the operator T has dense range. Here T is strictly cosingular if and only if T' is strictly singular.

In the present attempt we are concerned with inclusion operators $i : F \to E$ between BK-spaces. We seek to express strict singularity and strict cosingularity of i in terms of the β- and γ-dual operators i^β, i^γ. Some results of this kind, dealing with concrete duality instead of topological duality, have been obtained by Snyder in [21] and by the author in [12]. These investigations show that it is helpful to discuss, in this context, two sequence space properties, which are closely related with strict singularity and cosingularity, and which are familiar in sequence space theory.

In [21,22], Snyder considers the following property of BK-spaces E, F having $\Phi \subset F \subset E$. The relation $F < E$ is said to hold if, given any BK-space G containing Φ and satisfying $E = F + G$, one must have $G = E$. In [22, Lemma 3.1] it is proved that $F < E$ always implies strict cosingularity of $i : F \to E$, while the converse is true when either F or E has AD. Actually, some assumption of this kind on F or E is needed. This may be seen from the example $c \to \ell_\infty$. Indeed, this inclusion is strictly cosingular, (cf. [18]), but $c \not< \ell_\infty$. For the latter choose for G any topological complement of the sequence e containing Φ in ℓ_∞.

The following sequence space property, closely related to strict singularity, was discussed by Snyder [21,22] and the author [12]. Let X, Y be BK-spaces satisfying $\Phi \subset X \subset Y$. The inclusion $i : X \to Y$ is said to have the ***Meyer-König/Zeller property*** (noted MKZ) if, given any BK-space W having $Y \cap W \subset X$, $W \cap X$ must be closed in W. For the first time, this property was implicitely considered by Meyer-König and Zeller [9,10], who proved that $c_o \to \ell_\infty$ has the MKZ. As a technical tool, they used another sequence space property, called the ***gliding humps property***, which was further investigated among others in [21] and [12]. For various other examples of inclusions having MKZ or the gliding humps property, we refer to the references above. In [12] we proved the following

Lemma 6.1 *Let X, Y be BK-spaces having $\Phi \subset X \subset Y$. Then*

(1) If $Y = F^\beta$ for a BK-space F containing Φ, then strict singularity of $i : X \to Y$

implies the MKZ;

(2) If $X = E^\gamma, Y = F^\gamma$ for BK-spaces E, F having $\Phi \subset F \subset E$, then strict singularity and the MKZ for i are equivalent properties.

In general, strict singularity of $i : X \to Y$ is stronger than the MKZ or the gliding humps property. Indeed, the inclusion $c_o \to \ell_\infty$ is not strictly singular (for c_o is closed in ℓ_∞), but has the MKZ (cf. [21]). The following remark is even more substantial. MKZ does not imply strict singularity even when X is a β-dual and Y is a γ-dual. Indeed, take the inclusion $cs \to bs$. Then $cs = bv^\beta, bs = bv^\gamma$, the inclusion is not strictly singular, for cs is closed in bs, but it has the gliding humps property hence the MKZ (cf. [21]).

Theorem 6.2 *Let E, F be BK-spaces having $\Phi \subset F \subset E$. Suppose E has the Wilansky property $(\gamma - W1)$. Then if $i^\beta : E^\beta \to F^\beta$ or $i^\gamma : E^\gamma \to F^\gamma$ is strictly singular, the inclusion $i : F \to E$ is strictly cosingular.*

Proof. 1) First observe that strict singularity of i^γ implies strict singularity of i^β. Indeed, if i^β is not strictly singular, there exists an infinite dimensional linear subspace S of E^β which is closed in F^β. As F^β is closed in F^γ, we deduce that i^γ is not strictly singular. Hence it suffices to prove the statement in the case where i^β is assumed strictly singular.

2) We check the following property of the inclusion i, which is a modification of Snyder's property $<$. Given any BK-space G containing Φ such that $E = F + G$, G must be closed in E. Indeed, let G be of this kind. Then we have $E^\beta = F^\beta \cap G^\beta$. By Lemma 6.1, strict singularity of i^β implies the MKZ. This means that E^β is closed in G^β, hence by Lemma 2.2 we obtain $E^\gamma = G^\gamma$. Consequently, we may now apply the Wilansky property $(\gamma - W1)$, which implies that G is closed in E.

3) We next prove that the statement of 2) above holds for arbitrary BK-spaces G not necessarily containing Φ. We prove this using the reduction technique applied in [21]. Let G be any BK-space satisfying $E = F + G$. Let z be a sequence of entries $z_n \neq 0$ such that $z\ell_1 \subset E$ and the inclusion $z\ell_1 \to E$ is compact. Let $G_o = G + z\ell_1$. Then G_o is a BK-space containing Φ, hence part 2) guarantees that G_o is closed in E. Therefore $z\ell_1$ is compactly included in G_o. Now let $\{v^\alpha : \alpha \in I\}$ be a dense set of vectors in the unit ball of G_o. Find bounded sets $\{x^\alpha : \alpha \in I\}$ in G, $\{y^\alpha : \alpha \in I\}$ in $z\ell_1$ having $v^\alpha = x^\alpha + y^\alpha$. Now define operators $A, B, C : \ell_1(I) \to G_o$ by $A\lambda = \sum \lambda_\alpha x^\alpha, B\lambda = \sum \lambda_\alpha y^\alpha, C\lambda = \sum \lambda_\alpha v^\alpha$. Then C is an open mapping, i.e. it is surjective. Moreover, $A(\ell_1(I)) \subset G$, $B(\ell_1(I)) \subset z\ell_1$. So B is compact, hence using an argument from perturbation theory, the equality $A = C - B$ shows that $A(\ell_1(I))$ has finite codimension in G_o. So G has finite codimension in G_o, hence in particular is closed in G_o, and so is closed in E as well.

4) Let us now check that $i : F \to E$ is strictly cosingular. Let M be a closed subspace of E, let $q : E \to E/M$ be the quotient mapping, and suppose $q \circ i$ is surjective. We

have to prove that $Y = E/M$ is finite dimensional. Let L be any BK-space having $M \subset L \subset E$. Then we have $E = F + L$. Indeed, given any $x \in E$, we find $y \in F$ having $q(x) = q(i(y)) = q(y)$, so $x = y + (x - y)$ with $x - y \in \operatorname{Ker} q = M \subset L$. Consequently, by part 3), any such L must be closed in E. It remains to prove that this implies that M must have finite codimension in E.

5) Assume that Y is infinite dimensional. Then it is possible to select a linearly independent sequence (y^n) of vectors in Y having $\|y^n\| = 1$, which converges to 0 weakly. Now select a basic subsequence of (y^n) also denoted by (y^n). Define a linear operator $T : \ell_\infty \to Y$ by setting

$$T\lambda = \sum_{i=1}^{\infty} \frac{\lambda_i}{i^2} \cdot y^i$$

Observe that T is well-defined and continuous in view of $\|y^i\|_Y = 1$. Moreover it is injective since $(\frac{1}{i^2} y^i)$ is a basic sequence in Y. Let Z denote the image of ℓ_∞ under T with norm induced by ℓ_∞, i.e. $\|z\|_Z = \|\lambda\|_\infty$ in case $z = T\lambda$. Then Z is a Banach space. Observe that, consequently, Z is not a closed subspace of Y, for the sequence $(\frac{1}{n^2} y^n)$ has norm 1 in Z. Now let $L = q^{-1}(Z)$, then we have $M \subset L \subset E$, and L can be made into a BK-space by taking as a norm $\|x\| = \|x\|_E + \|q(x)\|_Z$. By part 4) above, L should be closed in E, but this is absurd. For let (x^n) be a bounded sequence in E having $q(x^n) = y^n$. Then (x^n) would be bounded in L, hence (y^n) had to be bounded in Z, which is not the case as $(\frac{1}{n^2} y^n)$ is bounded away from 0 in Z. This completes our argument. □

Remarks. 1) If the space F above is assumed AD, then the Wilansky property (W) for E is sufficient to give the statement of the Theorem. For in this case, any space G containing Φ and satisfying $E = F + G$ is automatically dense in E, so part 2) of the proof works with property (W).

2) Theorem 6.2 above admits a converse under more restrictive assumptions on the space E (cf. [12]). The statement is that strict cosingularity of i implies strict singularity of i^γ when E is a BK-AK-space such that the closure of Φ in E' has a separable topological complement. No special assumptions on the space F are needed.

Another converse result is obtained if F is assumed to have AB. In this case no restrictions on the space E are needed, but something additional has to be required for the mapping i.

Proposition 6.3 *Let E, F be BK-spaces having $\Phi \subset F \subset E$. Suppose F has AB and $i : F \to E$ is strictly cosingular. Suppose that either (i) F is separable and dense in E, or (ii) i is weakly compact. Then i^β and i^γ are strictly singular.*

Proof. Both conditions (i) and (ii) guarantee that the adjoint operator $i' : E' \to F'$ is strictly singular. In case (i) this is Snyder's result [22], in case (ii) this was proved in [18].

But now $i^\beta = i' \restriction E^\beta$ is strictly singular as a mapping $E^\beta \to F'$. As F has AB, F^β is closed in F', so i^β is as well strictly singular as a mapping $E^\beta \to F^\beta$. This ends the proof of the β-dual version. The γ-dual part is proved analogously. □

In the final part of this section we ask for a γ-dual version of Pełczyński's result stating that strict cosingularity of the adjoint T' implies strict singularity of T. Here the situation is less satisfactory.

Proposition 6.4 *Let E, F be BK-spaces having $\Phi \subset F \subset E$. Suppose F has AK and E has AB. Then strict cosingularity of $E^\gamma \to F^\gamma$ implies strict singularity of $F \to E$.*

Proof. Let E_o be the closure of Φ in $E^{\gamma\gamma}$. Then we have $F \subset E_o$ by the assumptions. Moreover, $E_o' = E^\gamma$, so the inclusion $i^\gamma : E^\gamma \to F^\gamma$ is just the restriction mapping $i' : E_o' \to F'$, which by assumption is strictly cosingular. Hence by Pełczyński's result, the inclusion $F \to E_o$ is strictly singular. As E has AB, we have $E_o \subset E$ by Lemma 3.1, so $F \to E$ is strictly singular. This ends the proof. □

Remark. Notice that Proposition 6.4 is rather a general scheme than a theorem. Using the very same reasoning, we might obtain a result for any pair $\mathbf{P}, \mathbf{P}'$ of properties of linear operators such that $\mathbf{P}'$ for T' implies $\mathbf{P}$ for T. More precisely, under the above assumptions on E, F we derive the statement that $\mathbf{P}'$ for i^γ implies $\mathbf{P}$ for i, at least when property $\mathbf{P}$ stands for a right ideal of operators.

In the following we prove a somewhat more involved result. Relaxing the assumptions on the spaces E, F, which are quite restrictive in Proposition 6.4 above, we obtain a weaker conclusion on $F \to E$. First let us recall a definition.

A sequence (z^n) of vectors $z^n \neq 0$ in Φ is called a block sequence if there exists a strictly increasing sequence (k_n) of indices such that $z_k^n \neq 0$ only for $k_{n-1} < k \leq k_n$.

Let E, F be BK-spaces having $\Phi \subset F \subset E$. The inclusion $F \to E$ is said to have the *gliding humps property* if, given any block sequence (z^n) having $\|z^n\|_F = 1$, there exists a sequence $\lambda \in \ell_\infty$ such that

$$z = \sum_{i=1}^{\infty} \lambda_i z^i \in E \setminus F,$$

where the sum is understood in the pointwise sense. For details concerning this notion see [21,12,13]. In [12] we proved that strict singularity of $F \to E$ implies the gliding humps property if E has AB.

With these definitions we may now state the following

Theorem 6.5 *Let E, F be BK-AB-spaces having $\Phi \subset F \subset E$. Suppose F does not contain an isomorphic copy of ℓ_∞. Then strict cosingularity of i^γ implies that $F \to E$ has the gliding humps property.*

Proof. 1) Let (z^n) be a block sequence having $\|z^n\|_F = 1$. Let L be defined by

$$L = \Big\{ \sum_{i=1}^{\infty} \lambda_i \cdot z^i : \lambda \in \ell_\infty \Big\}.$$

Similarly, we define L_o as the subspace of L consisting of those $z \in L$ having $\lambda \in c_o$. Then L, L_o are BK-spaces with the norm $\|z\|_L = \|\lambda\|_\infty$, hence in particular $L \cong \ell_\infty, L_o \cong c_o$. We have to prove that $E \cap L \not\subset F$. Assume the contrary, i.e. $E \cap L \subset F$. Let G be the BK-space $G = F + L$, then we have $E \cap G = F$. We claim that G has AB. Indeed, let $x \in G$, $x = y + z$, with $y \in F$, $z \in L, z = \sum \lambda_i z^i$. Let $k \in \mathbf{N}$, and choose n having $k_{n-1} < k \leq k_n$. Then we have

$$P_k x = P_k y + P_{k_{n-1}} z + \lambda_n \cdot P_k z^n.$$

This implies

$$\begin{aligned} \|P_k x\|_G &\leq \|P_k y + \lambda_n \cdot P_k z^n\|_F + \|P_{k_{n-1}} z\|_L \\ &\leq \|P_k y\|_F + \|\lambda\|_\infty \cdot \|P_k z^n\|_F + \|P_{k_{n-1}} z\|_L \\ &\leq C \cdot \|y\|_F + C \cdot \|\lambda\|_\infty \cdot \|z^n\|_F + \|\lambda\|_\infty \\ &\leq (C+1) \cdot (\|y\|_F + \|z\|_L), \end{aligned}$$

where $C > 0$ is a constant having $\|P_k y\|_F \leq C \cdot \|y\|_F$ for all $y \in F, k \in \mathbf{N}$. By the definition of the norm on G (cf. [6]), this implies

$$\|P_k x\|_G \leq (C+1) \cdot \|x\|_G.$$

Hence G has AB.

2) Now we apply [6, Satz 2.3(c)], which implies $E^\gamma + G^\gamma = F^\gamma$. As $E^\gamma \to F^\gamma$ is assumed strictly cosingular, [22, Theorem 3.2] gives that G^γ has finite codimension in F^γ, and this means $G^\gamma = F^\gamma$ by Lemma 2.3. Consequently, we have $L \subset F^{\gamma\gamma}$. This implies $L_o \subset F_o$, where F_o denotes the closure of Φ in $F^{\gamma\gamma}$. Hence $L_o \subset F$, for F has AB.

3) Next observe that L_o, when endowed with the norm $\|\cdot\|_L$, is a closed subspace of F. Indeed, let $z \in L_o$, $z = \sum \lambda_i z^i$. Then we have

$$\begin{aligned} |\lambda_n| &= \|\lambda_n z^n\|_F \\ &= \Big\| \sum_{i=1}^{n} \lambda_i z^i - \sum_{i=1}^{n-1} \lambda_i z^i \Big\|_F \\ &= \|P_{k_n} z - P_{k_{n-1}} z\|_F \\ &\leq 2C \cdot \|z\|_F, \end{aligned}$$

proving $\|z\|_L \leq 2C \cdot \|z\|_F$. Hence L_o is closed in F.

4) Next observe that $E^\gamma = E_o', F^\gamma = F_o'$, so by the result of Pełczyński [18], strict cosingularity of i^γ implies that $F_o \to E_o$ is strictly singular, hence in view of the fact that E has AB, the inclusion $F_o \to E$ is strictly singular. This means that L_o, being infinite dimensional, cannot be closed in E. Consequently the norms $\|z^n\|_E$ of the vectors z^n may not be bounded away from 0 in E, for otherwise the argument used in part 3) above would show L_o as a closed subspace of E. Let us therefore choose a sequence (n_k) of indices such that $\sum_k \|z^{n_k}\|_E < \infty$.

5) Let M be the subspace of L consisting of all $z = \sum_k \lambda_{n_k} z^{n_k}$, $\lambda \in \ell_\infty$. Then $M \subset E$ by the above choice of the sequence (n_k). So our assumption $E \cap L \subset F$ implies $M \subset F$. Now the argument from part 3) shows that M must be closed in F. As $M \cong \ell_\infty$, we obtain a contradiction with the fact that F does not contain a copy of ℓ_∞. This ends the proof. □

Remark. The AB assumption on E may not be omitted here, even when F has AK. This may be seen from the main example in section 7.

7 The Main Example

In this section we present an example which was quoted at some place or other during the previous sections.

Let $E = \{x \in \ell_1 : ((2n)^2 x_{2n} - (2n+1)^2 x_{2n+1}) \in c_o\}$. Then E is a separable BK-space under its natural norm. Obviously, E is a proper dense subspace of ℓ_1. Nevertheless, we have the following

Claim 1. $E^\beta = \ell_\infty$.

Indeed, let $y \notin \ell_\infty$, and choose a sequence (n_i) such that $|y_{n_i}| \geq i^3$. We may assume that all n_i are either even or odd. Assume they are even. Define a sequence x by

$$x_{n_i} = \frac{1}{i^2}, \quad x_{n_i+1} = \Big(\frac{n_i}{n_i+1}\Big)^2 x_{n_i}, \quad x_n = 0 \qquad \text{otherwise}$$

Clearly $x \in \ell_1$, and $(2n)^2 x_{2n} - (2n+1)^2 x_{2n+1} = 0$. Hence $x \in E$. But $xy \notin cs$, for $x_{n_i} y_{n_i} \nrightarrow 0$. This proves the claim in the case where the n_i are even. A similar argument works when they are assumed odd.

Let $F = \{x \in \omega : (n^2 x_n) \in c_o\}$. Then F is a BK-AK-space which is isomorphic with c_o and is contained in E. Let us now establish several facts on the inclusions $F \to E$ and $E^\beta \to F^\beta$.

Claim 2. *$E^\beta \to F^\beta$ is compact.*

Indeed, this follows from Schauder's Theorem combined with the fact that the inclusion $F \to \ell_1$ is compact, the latter in view of $F \cong c_o$ (cf. [8, §42]).

Claim 3. *$F \to E$ is not compact.*

Let $x \in \omega$ be the sequence $x_n = \frac{(-1)^n}{n^2}$, and consider the sequence $(P_n x)$ of sections of x. Clearly $\|P_n x\|_F = 1$, so compactness of the inclusion $F \to E$ would imply the existence of a convergent subsequence in E. Clearly, the only possible choice for a limit of a convergent subsequence could be x. But notice that $x \notin E$ in view of $(2n)^2 x_{2n} - (2n+1)^2 x_{2n+1} = 2$. This proves the claim. As an immediate consequence we obtain the next

Claim 4. *$F \to E$ is not weakly compact.*

Indeed, this follows from the fact that $F \cong c_o$ and that weakly compact operators with source space c_o are compact (cf. [8, §42]).

Claim 5. *$F \to E$ is not strictly singular.*

Notice that $F \cong C(S)$ for a space $C(S)$ of continuous functions on a compact Hausdorff space. Hence Pełczyński's result [18, Theorem 1] tells that strict singularity of the inclusion would imply weak compactness here, and the latter is not valid by Claim 4.

Claim 6. *$F \to E$ does not have the MKZ.*

Indeed, let W be the BK-space defined by

$$W = \{x \in \ell_1 : ((2n)^2 x_{2n} + (2n+1)^2 x_{2n+1}) \in c_o\}.$$

Then we have $W \cap E = F$. Indeed, a sequence $x \in W \cap E$ satisfies

$$(2n)^2 x_{2n} - (2n+1)^2 x_{2n+1} \to 0,$$
$$(2n)^2 x_{2n} + (2n+1)^2 x_{2n+1} \to 0,$$

giving $(2n)^2 x_{2n} \to 0$ and $(2n+1)^2 x_{2n+1} \to 0$, hence $n^2 x_n \to 0$.

But F is not closed in W, as it should if $F \to E$ had the MKZ. Indeed, take the sequence x having $x_n = \frac{(-1)^n}{n^2}$, then $P_{2n}x \to x$ in W, but $P_{2n}x \not\to x$ in F, for $x \notin F$.

Claim 7. *$F \to E$ does not have the gliding humps property.*

Indeed, the gliding humps property would imply MKZ here by [21, Theorem 1], contrary to Claim 6.

Completing our list of abhorrend properties of the inclusion $F \to E$, we add the following

Claim 8. *$F \to E$ is not strictly cosingular.*

As F has AK, it suffices to show that $F \not< E$, for strict cosingularity and Snyder's property are equivalent, then.

Let $G = \{x \in \ell_1 : \sum_n \mid (2n)^2 x_{2n} - (\overset{\circ}{2}n+1)^2 x_{2n+1} \mid < \infty\}$, then G is a proper BK-subspace of E containing Φ. We prove that $E = F + G$, which shows $F \not< E$.

Let $x \in E$ be fixed. Define $y \in \omega$ by

$$y_{2n} = x_{2n} - \Big(\frac{2n+1}{2n}\Big)^2 x_{2n+1}, \qquad y_{2n+1} = 0.$$

Then we have $(2n)^2 y_{2n} = (2n)^2 x_{2n} - (2n+1)^2 x_{2n+1} \to 0$, so $y \in F$. Let $z = x - y$. We have to check that $z \in G$. Observe that

$$z_1 = x_1, \qquad z_{2n} = \Big(\frac{2n+1}{2n}\Big)^2 x_{2n+1}, \qquad z_{2n+1} = x_{2n+1},$$

so actually $(2n)^2 z_{2n} - (2n+1)^2 z_{2n+1} = 0$, proving $z \in G$. This proves our claim.

Clearly, Claim 8 implies Claim 3, for compact operators are strictly cosingular. But a direct proof of Claim 3 is fairly easy.

References

[1] G. Bennett: A new class of sequence spaces with applications in summability theory. J. Reine Angew. Math. 266 (1974), 49 - 75.

[2] G. Bennett: Sequence spaces with small β-duals. Math. Z. 194 (1987), 321 - 329.

[3] J.W. Brace, R.R. Kneece: Approximation of strictly singular and strictly cosingular operators using nonstandard analysis. Trans. Amer. Math. Soc. 168 (1972), 483ff.

[4] N. Dunford, J.T. Schwartz: Linear operators I. Interscience Publishers, New York, 1958.

[5] D.J.H. Garling: The β- and γ-duality of sequence spaces. Proc. Camb. Phil. Soc. 63 (1967), 963 - 981.

[6] G. Goes: Summen von FK-Räumen. Funktionale Abschnittskonvergenz und Umkehrsätze. Tôhoku Math. J. 26 (1974), 487 - 504.

[7] T. Kato: Perturbation theory for nullity, deficiency and other quantities of linear operators. Journ. D'Analyse Math. 6 (1958), 261 - 322.

[8] G. Köthe: Topological Vector Spaces I,II. Springer Verlag 1969, 1979.

[9] W. Meyer-König, K. Zeller: Lückenumkehrsätze und Lückenperfektheit. Math. Z. 66 (1956), 203 - 224.

[10] W. Meyer-König, K. Zeller: FK-Räume und Lückenperfektheit. Math. Z. 78 (1962), 143 - 148.

[11] D. Noll: Sequence spaces with separable γ-duals. Archiv der Math. 54 (1990), 73 - 83.

[12] D. Noll: Strict (co)singularity of inclusion operators between γ-spaces and Meyer-König Zeller type theorems. Illinois J. Math. *to appear*.

[13] D. Noll: Sequential completeness and spaces with the gliding humps property. Manuscripta Math. 66 (1990), 237 - 252.

[14] D. Noll, W. Stadler: Sliding hump technique and spaces with the Wilansky property. Proc. Amer. Math. Soc. 105 (1989), 903 - 910.

[15] D. Noll, W. Stadler: Weakening Tauberian conditions for summability methods. Analysis 9 (1989), 41 - 53.

[16] D. Noll, W. Stadler: Zerlegungen von Wachstumsbereichen und Wirkfeldern für die Verfahren bewichteter Mittel. Manuscripta Math. 60 (1988), 197 - 209.

[17] D. Noll, W. Stadler: Abstract sliding hump technique and characterization of barrelled spaces. Studia Math. XCIV (1989), 103 - 120.

[18] A. Pełczyński: On strictly singular and strictly cosingular operators I,II. Bull Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. 13 (1965), 31 - 41.

[19] M. Schaffer, A.K. Snyder: Properties of sequence spaces in which ℓ_1 is weakly compactly embedded. Math. Z. 192 (1986), 569 - 574.

[20] A.K. Snyder: A property of the embedding of c_o in ℓ_∞. Proc. Amer. Math. Soc. 97 (1986), 59 - 60.

[21] A.K. Snyder: An embedding property of sequence spaces related to Meyer-König and Zeller type theorems. Indiana Univ. Math. J. 35 (1986), 669 - 679.

[22] A.K. Snyder: Strictly (co)singular operators, the Meyer-König Zeller property, and sums of Banach spaces. Illinois J. Math. 33 (1989), 93 - 102.

[23] W. Stadler: Zu einer Frage von Wilansky. Archiv der Math. 48 (1987), 149 - 152.

[24] A. Wilansky: Summability through Functional Analysis. North Holland, New York 1984.

Universität Stuttgart, Mathematisches Institut B, Pfaffenwaldring 57, 7000 Stuttgart 80, BR Deutschland

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