Consistency of a Nonlinear Deconvolution Method with Applications to Image Restoration [1]

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Abstract

We prove deterministic and stochastic consistency of a nonlinear image restoration technique based on a maximum entropy model. Practical aspects of the method are also addressed, and we conclude with some numerical experiments.

Key words: Image restoration, consistency of a non-linear regression estimate, non-linear deconvolution, deblurring, maximum entropy method.

1 Motivation

Picture restoration deals with images that have been recorded in the presence of various sources of degradation. Some types of degradations, called point degradations, affect only the gray levels of the individual pixels. Other types involving blur are called spatial degradations. For example, in aerial reconnaissance, astronomy and remote sensing, the pictures are degraded by atmospheric turbulence, optical system aberrations, and/or object and camera motions. Electron micrographs are often degraded by the spherical aberrations of the electron lens, and medical radiographs are of low resolution and contrast, (see e.g. [20], [1] for more information on these).

Given an ideal image $u(x)$ over a region $\Omega \subset \mathbb{R}^2$ and the corresponding degraded image $v(x)$, we will assume that u and v are related by

$$v(x) = \int_\Omega q(x,y)\, u(y)\, dy + e(x), \quad x \in \Omega, \tag{1.1}$$

where $q(x,y)$ represents the blurring and $e(x)$ the random pixel noise.

The assumption that the observed image $v(x)$ is a linear function of the original $u(x)$ may be questioned in a variety of situations. For example, in the restoration of photographic

[1] Advances in Mathematical Sciences and Applications, Vol. 7, No. 2 (1997), pp. 789 – 808

images, what is recorded is usually a nonlinear function of $u(x)$. However, in practice, the nonlinear characteristics of the exposure mapping are often known, and it is then possible to perform an a priori gray level correction which finally leads to a linear model of type (1.1); cf. [21].

Similarly, the additivity of the signal independent noise may be challenged. Many noise sources are in fact additive. However, if followed by some nonlinear transformation, the additivity will only be a valid assumption over a small dynamic range. In some cases, the appropriate noise model is multiplicative and may be converted into an additive form by applying a logarithmic transform. In summary, both hypotheses, the additivity of the noise source, and the linearity of the blur are commonly accepted in image restoration since they make the problem mathematically tractable.

If the degradation caused by the blur is spatially invariant, and this is usually met at least over large parts of the region, the blur takes the convolutional form $q(x,y) = q(x-y)$, and $q(x)$ is then called the point-spread-function (PSF). In the following we will limit our consideration to the deconvolution problem: Based on the knowledge of $q(x)$ and some statistical a priori knowledge about the noise, and assuming the model

$$v(x) = (q * u)(x) + e(x) = \int_\Omega q(x-y)\, u(y)\, dy + e(x) \tag{1.2}$$

we wish to recover the unknown $u(x)$ from its degraded version $v(x)$.

The hypothesis that the blur $q(x)$ is known is clearly special. In many cases, the physical phenomenon underlying the degradation can be used to determine $q(x)$, but in other situations one might have to perform an a posteriori determination of the PSF by directly inspecting the degraded image $v(x)$, a procedure which requires individual treatment of the images under considerations. In the present investigation, we will always assume that the PSF is known.

One classical approach to the deconvolution problem (1.2) is based on inverse Wiener or Kalman filtering, cf. [2, 17]. These methods, however, require some a priori knowledge about the power spectra of the unknown signal $u(x)$ and noise $e(x)$ resp. their cross-spectrum, a hypothesis which often appears to be artificial. More realistically, all that may be known about the statistics of the model (1.2) might be the noise variance σ_e^2, which in practice is determined by inspecting parts of the degraded image $v(x)$ with a relatively homogeneous gray tone.

Assume in the following that $\Omega = [0,1] \times [0,1]$, and suppose that the degraded image $v(x)$ has been sampled at the nodes x_{ij} of a rectangular grid Ω_h with mesh size $h > 0$:

$$v_{ij} = (q * u)_{ij} + e_{ij} = (q * u)(x_{ij}) + e(x_{ij}), \quad 0 \le i, j \le N-1.$$

Suppose $q \in L^\infty(\Omega)$, and let the operator $Q_h : L^1(\Omega) \to \mathbb{R}^{N \times N}$ and the vector v_h be defined respectively by

$$(Q_h u)_{ij} = (q * u)_{ij} = (q * u)(x_{ij}), \qquad (v_h)_{ij} = v(x_{ij}) = v_{ij}.$$

Under the assumption that the sequence e_{ij} satisfies the strong law of large numbers we obtain the estimate

$$\|v_h - Q_h u\|_2^2 = \|e_h\|_2^2 = \sum_{ij} e_{ij}^2 \approx N^2 \sigma_e^2, \tag{1.3}$$

where $N^2 = (1 + 1/h)^2$ is the number of pixels in Ω_h. It is then reasonable to seek for a reconstruction of $u(x)$ among the functions satisfying (1.3), and this has been suggested in the engineering literature in a variety of papers (see [3], [15], or [14] among others).

Since there are clearly many solutions of (1.3), we select the one which minimizes some previously chosen performance index like an energy functional or an entropy/information measure. This leads to the optimization program

$$(P_1) \quad \begin{array}{rl} \text{minimize} & \mathcal{I}(u) = \int_\Omega \phi(u(x), Du(x))\, dx \\ \text{subject to} & \|Q_h u - v_h\|_2 \leq N\sigma_e =: N\epsilon, \\ & u(x) \geq 0, \ \int_\Omega u(x)\, dx = 1. \end{array}$$

The Maximum Entropy approach is part of this scheme when we choose the performance index to be the Boltzmann-Shannon entropy expression

$$\phi(x,p) = \phi(x) = \begin{cases} x \log x & x \geq 0 \\ +\infty & x < 0 \end{cases}.$$

Alternative choices of objectives $\phi(x,p)$ which control derivative values of the unknown $u(x)$ are the energy integral $\mathcal{I}(u) = \int_\Omega |Du(x)|^2\, dx$, or the higher order index $\mathcal{I}(u) = \int_\Omega |\Delta u(x)|^2\, dx$ considered by Hunt [16]. As a further possibility we mention Fisher's information $\mathcal{I}(u) = \int_\Omega \frac{|Du(x)|^2}{u(x)}\, dx$, which was successfully used in [8, 9, 10] for power spectrum estimation problems.

An alternative approach towards solving the deconvolution problem (1.2) is based on regression analysis. On applying an operator of local averaging type to the data v_h, like for instance a kernel estimator or a nearest neighbour estimator, provides an estimate $\tilde{v}_h$ for the unknown function $q * u$. The latter may then – with the error $\tilde{e}_h$ significantly reduced in size – be used to get a restoration u_h by performing, say, a least square inversion of the linear system $(q * u)_h = \tilde{v}_h$. This method clearly requires a trade-off between the smoothing effects needed to reduce the noise, and the additional blur thereby introduced. In the presence of a sizable noise component this limits the outlined approach.

The method we propose is to combine the optimization point of view with regression analysis. More precisely, we suggest to apply the scheme (P_1) after an a priori smoothing of the data has been performed. The latter may be done e.g. by applying a kernel operator

$\tilde{v}_h = k * v_h$ to the sampled data v_h, which means that (P_1) is now replaced by the following scheme:

$$(P_2)\quad \begin{array}{ll} \text{minimize} & \mathcal{I}(u) = \int_\Omega \phi(u(x), Du(x))\,dx \\ \text{subject to} & \|k * Q_h u - k * v_h\|_2 \le N\tilde{\epsilon} \\ & u(x) \ge 0,\ \int_\Omega u(x)\,dx = 1. \end{array}$$

The tolerance $\tilde{\epsilon}$ may be determined through the same type of argument based on the law of large numbers: $\|k * e_h\|_2^2 = \|\tilde{e}_h\|_2^2 \approx N^2\sigma_{\tilde{e}}^2 = N^2(\sum_{ij} k_{ij}^2)\sigma_e^2 =: N\tilde{\epsilon}^2$, (cf. Section 5).

It is clear that a serious assessment of the virtues of the models $(P_1), (P_2)$ in solving real life deconvolution problems could only be based on an analysis of their numerical performance. This issue has been addressed in our paper [18], cf. also [3, 14, 11, 15], and we present a variety of experiments with the model (P_2). Nevertheless, asking for theoretical arguments in support of these models is still worthwhile. Here we shall do this by proving deterministic and stochastic consistency of the model (P_2) for an appropriate choice of the model parameter $\tilde{\epsilon}$. It turns out that such analysis has something to offer even for practice. Namely, our arguments suggest that the choice $\epsilon = \sigma_e$ resp. $\tilde{\epsilon} = \sigma_{\tilde{e}}$ is not the final answer. In fact a slightly larger tolerance level is needed (see Section 5). The latter then allows for a consistency result (Theorem 5.1).

2 Duality

In this section we recall some known facts needed to analyze program (P_1) with the Boltzmann-Shannon entropy as performance index:

$$(P_1)\quad \begin{array}{ll} \text{minimize} & \mathcal{I}(u) = \int_\Omega u(x)\log u(x)\,dx \\ \text{subject to} & \|Q_h u - v_h\|_p \le h^{-1}\epsilon_h, \\ & u(x) \ge 0,\ \int_\Omega u(x)\,dx = 1, \end{array}$$

where $\|\cdot\|_p$ with $1 \le p \le \infty$ denotes any fixed $p-$norm. Some basic references on Maximum Entropy type models are [5, 6, 7, 13]. In particular, for the following, the reference [4] is extremely useful.

The concave dual program associated with (P_1) is

$$(D_1)\quad \begin{array}{ll} \text{maximize} & \mathcal{J}(\lambda, \mu) = \sum_{ij} \lambda_{ij} v_{ij} + \mu - \int_\Omega \exp\{(Q_h^*\lambda)(x) + \mu - 1\}\,dx - h^{-1}\epsilon_h\|\lambda\|_{p'} \\ \text{subject to} & \lambda \in \mathbb{R}^{N\times N},\ \mu \in \mathbb{R}, \end{array}$$

where $1/p + 1/p' = 1$, and where the adjoint $Q_h^* : \mathbb{R}^{N\times N} \to L^\infty(\Omega)$ is defined by $Q_h^*(\lambda) = \sum_{ij} \lambda_{ij} q(x_{ij} - \cdot) =: \sum_{ij} \lambda_{ij} q_{ij}$.

Let the values of the primal and dual programs be $V(P_1)$ resp. $V(D_1)$. It is a standard argument in convexity that under a mild constraint qualification hypothesis (CQ), the primal and dual programs are equivalent in the sense that their values coincide, and a complementary slackness type relation tells that the optimal solution $\bar{u}_h$ of (P_1) may be represented in terms of the solution $(\bar{\lambda}_h, \bar{\mu}_h)$ of (D_1) through a return formula, which in our case is:

$$\bar{u}_h(x) = \exp\{Q_h^*(\bar{\lambda}_h)(x) + \bar{\mu}_h - 1\} = \exp\Big\{\sum_{ij} \bar{\lambda}_{ij} q(x_{ij} - x) + \bar{\mu}_h - 1\Big\}. \tag{2.1}$$

A constraint qualification sufficient to guarantee this equivalence is the following (cf. [6, 4, 7]):

$$(CQ) \qquad \begin{array}{l} \text{There exists } \hat{u} \text{ piecewise continuous, } \hat{u} > 0, \\ \displaystyle\int_\Omega \hat{u}(x)\,dx = 1, \text{ such that } \|Q_h\hat{u} - v_h\|_p \le h^{-1}\epsilon_h. \end{array}$$

3 Deterministic Estimates

In this section we obtain deterministic estimates for the program (P_1) with Boltzmann-Shannon entropy objective. We mention that a convergence analysis for a deterministic program with Boltzmann-Shannon entropy applied to Fourier type inversion problems has been obtained by the authors of [4]. Some of their technical results will enter into the present reasoning.

In the following, let the unknown signal we are trying to reconstruct be $\bar{u}(x) \ge 0$ with $\int_\Omega \bar{u}(x)\,dx = 1$. In order to exclude trivial situations, we assume that $\bar{u}$ is piecewise continuous with $\bar{u} > 0$. For a given $q \in L^\infty(\Omega)$, let the true values of $\bar{v} = q * \bar{u}$ at the grid points x_{ij} $(0 \le i, j \le N-1)$ of Ω_h be $\bar{v}_h$, where

$$\bar{v}_{ij} := (q * \bar{u})_{ij} = (q * \bar{u})(x_{ij}) = \int_\Omega q(x_{ij} - y)\bar{u}(y)\,dy. \tag{3.1}$$

Suppose the data v_h satisfy $\|\bar{v}_h - v_h\|_p \le h^{-1}\epsilon_h$. Then the constraint qualification hypothesis (CQ) is certainly satisfied (with $\bar{u}$ ranging in the place of $\hat{u}$).

For fixed $h > 0$, $K_h > 0$ and $1 \le p \le \infty$ let us consider the approximation constant

$$E_h(u, q, K_h) := \inf\{\|u - \sum_{i,j=0}^{N-1} \lambda_{ij} q(x_{ij} - \cdot)\|_\infty : \lambda \in \mathbb{R}^{N\times N},\ \|\lambda\|_{p'} \le K_h\}. \tag{3.2}$$

Our first estimate involving the constants K_h and E_h relates the values of (P_1) resp. (D_1) to $\mathcal{I}(\bar{u})$.

Lemma 3.1 *Let* $E_h := E_h(1 + \log \bar{u}, q, K_h)$. *Suppose* $\|\bar{v}_h - v_h\|_p \le h^{-1}\epsilon_h$. *Then*

$$\mathcal{I}(\bar{u}) \ge V(P_1) = V(D_1) \ge \mathcal{I}(\bar{u}) - \frac{1}{2}E_h^2 \exp\{E_h\} - 2h^{-1}\epsilon_h K_h. \tag{3.3}$$

Proof. The assumption $\|\bar{v}_h - v_h\|_p \le h^{-1}\epsilon_h$ guarantees that $\bar{u}$ is feasible for (P_1), and this implies the first inequality. As observed above, the feasibility of $\bar{u}$ also implies the fact that the constraint qualification (CQ) is satisfied, and this proves the dual equivalence $V(P_1) = V(D_1)$. It remains to prove the second inequality.

Choose λ such that the infimum (3.2) (with $u = 1 + \log \bar{u}$) is attained. Now use the estimate

$$\exp\{u\} \le 1 + u + \frac{1}{2}\beta^2 \exp\{\beta\} \qquad \text{for all } |u| \le \beta \tag{3.4}$$

which is verified using Taylor's formula (see [4]). Then applying (3.4) gives

$$\exp\{\sum_{ij} \lambda_{ij} q_{ij}(x) - 1 - \log \bar{u}(x)\} \le \sum_{ij} \lambda_{ij} q_{ij}(x) - \log \bar{u}(x) + \frac{1}{2}E_h^2 \exp\{E_h\}, \tag{3.5}$$

where $q_{ij} = q(x_{ij} - \cdot)$. Multiplying (3.5) by $\bar{u}(x) > 0$ and integrating gives

$$\int_\Omega \exp\{\sum_{ij} \lambda_{ij} q_{ij}(x) - 1\}\,dx - \sum_{ij} \lambda_{ij} \bar{v}_{ij} \le -\mathcal{I}(\bar{u}) + \frac{1}{2}E_h^2 \exp\{E_h\} \int_\Omega \bar{u}(x)\,dx. \tag{3.6}$$

By the definition of (D_1), and using (3.6), we have

$$\begin{aligned} V(D_1) \ge \mathcal{J}(\lambda, 0) &= -\int_\Omega \exp\{\sum_{ij} \lambda_{ij} q_{ij}(x) - 1\}\,dx + \sum_{ij} \lambda_{ij} v_{ij} - h^{-1}\epsilon_h \|\lambda\|_{p'} \\ &\ge \mathcal{I}(\bar{u}) - \frac{1}{2}E_h^2 \exp\{E_h\} - h^{-1}\epsilon_h \|\lambda\|_{p'} - \sum_{ij} \lambda_{ij}(\bar{v}_{ij} - v_{ij}). \end{aligned}$$

Now the last term can be estimated by

$$\Big|\sum_{ij} \lambda_{ij}(\bar{v}_{ij} - v_{ij})\Big| \le \|\lambda\|_{p'} \|\bar{v}_h - v_h\|_p \le h^{-1}\epsilon_h \|\lambda\|_{p'} \le h^{-1}\epsilon_h K_h,$$

which gives the second inequality in (3.3). □

With the help of Lemma 3.1, we are now in the position to give a first L^1-estimate for the approximation error $\|\bar{u} - \bar{u}_h\|_1$.

Proposition 3.2 *As above let* $E_h := E_h(1 + \log \bar{u}, q, K_h)$. *Suppose the data* v_h *of* (P_1) *satisfy* $\|v_h - \bar{v}_h\|_p \leq h^{-1}\epsilon_h$. *Then we have the following estimate:*

$$\|\bar{u} - \bar{u}_h\|_1^2 \leq E_h^2 e^{E_h} + 4K_h h^{-1}\epsilon_h. \tag{3.7}$$

Proof. Applying Proposition 4.6 in [4] gives

$$\int_\Omega \bar{u} \log \bar{u}_h \, dx \geq \int_\Omega \bar{u}_h \log \bar{u}_h \, dx = V(P_1). \tag{3.8}$$

Next observe that the estimate

$$|\exp\{v\} - \exp\{w\}| \leq \beta(1 + \exp\{\beta\}\beta/2)\exp\{w\} \qquad \text{for all } |v - w| \leq \beta, \tag{3.9}$$

which is Lemma 4.4(b) in [4], leads to

$$\frac{1}{2}\|\bar{u} - \bar{u}_h\|_1^2 \leq \int_\Omega \bar{u} \log(\bar{u}/\bar{u}_h) \, dx, \tag{3.10}$$

with the argument given in [4, §4]. Combining (3.8) and (3.10) with the estimate (3.3) implies the desired (3.7). □

Remark. Proposition 3.1 gives the clue to proving an L^1-norm convergence result. This requires two stages. As a first step we provide a choice of the constant $K_h > 0$ such that $E_h = E_h(1 + \log \bar{u}, q, K_h) \to 0$ as $h \to 0$. The second step will then be to adjust the model parameter ϵ_h in such a way that $K_h h^{-1}\epsilon_h \to 0$.

4 Interpolation Theorem

In this section we start upon our first stage mentioned above. This is achieved by the following result. Here we fix the notation q^+ for the adjoint kernel $q^+(x) = q(-x)$.

Theorem 4.1 *Let* $u \in H^t(\Omega)$ *for some* $t \geq 2$ *and suppose that* u *may be represented as* $u = q^+ * w$ *for some* $w \in L^{p'}(\Omega)$. *Then there exists a constant* c_t *depending only on* t *such that, with the choice* $K_h = c_t\|w\|_{p'}h^{(2p'-2)/p'} = c_t\|w\|_{p'}h^{2/p}$, *we obtain the estimate*

$$E_h(u, q, K_h) = \mathcal{O}(h^{t-2}). \tag{4.1}$$

Proof. As before let Ω_h be the rectangular grid with mesh $h > 0$. Let $\mathcal{T}_h^t$ be a triangulization of Ω consisting of congruent elements T having their corners on grid points from Ω_h, with each T containing a total of $t(t + 2)/2$ grid points. Take for instance triangles with hypothenuse pointing north-west to south-east. Here we may for simplicity assume that $t - 1$ divides $1/h$, so that Ω is covered by the elements in $\mathcal{T}_h^t$ with no overlap. If this is not the case, we may use an extension operator for Ω which carries the situation to a somewhat larger square where this extra requirement is satisfied.

Consider now the $\mathcal{C}^0$-finite elements on $\mathcal{T}_h^t$ with piecewise polynomials of degree $\leq t-1$ on each element $T \in \mathcal{T}_h^t$. Let $\mathcal{J}_h^t$ be the associated interpolation operator specified by the $t(t+1)/2$ grid points on each element. Then the Bramble-Hilbert Lemma (cf. [12]) implies

$$\|f - \mathcal{J}_h^t f\|_{H^2(\Omega)} \leq Ch^{t-2}\|f\|_{H^t(\Omega)} \tag{4.2}$$

for $f \in H^t(\Omega)$. The Sobolev embedding theorem for dimension 2 gives $H^2(\Omega) \subset \mathcal{C}(\bar{\Omega})$, and therefore implies

$$\|f - \mathcal{J}_h^t(f)\|_\infty \leq C'h^{t-2}\|f\|_{H^t(\Omega)} = \mathcal{O}(h^{t-2}). \tag{4.3}$$

Now let the elements T of $\mathcal{T}_h^t$ be labelled by their respective corners x_{rs} sited at the 90-degree angle, $(r,s) \in \Gamma_h^t \subset \Gamma_h = \{(i,j) : 0 \leq i,j \leq N-1\}$, say. There are two types of triangles. Let the elements pointing upward be noted T_{rs}^+, the elements pointing downward T_{rs}^-. For a fixed $(r,s) \in \Gamma_h^t$, let the $t(t+1)/2$ knots in $T_{rs}^\pm$ be labelled $x_{r\pm\alpha_i,s\pm\beta_i}$, $i = 1,2,\ldots,t(t+1)/2$. The interpolation operator has the following explicit form. For $y \in T_{rs}^\pm$:

$$(\mathcal{J}_h^t f)(y) = \sum_{i=1}^{t(t+1)/2} f(x_{r\pm\alpha_i,s\pm\beta_i})p_{rsi}(y) \tag{4.4}$$

for certain polynomials p_{rsi} of degree $\leq t-1$ depending only on r,s,i and h (resp. $N = 1+1/h$). Since the elements of $\mathcal{T}_h^t$ are congruent triangles, it follows from inspecting the situation for the reference element that there exists a constant C_t depending only on t and the domain Ω such that, for all r,s,i and $h>0$ the estimate

$$\left|\int_{T_{rs}^\pm} p_{rsi}(x)\,dx\right| \leq C_t h^2 \tag{4.5}$$

is satisfied.

For fixed $x \in \Omega$, let f be defined as $f(y) = w(y)q(y-x)$. Then integrating (4.3) implies

$$(q^+ * w)(x) = \int_\Omega f(y)\,dy = \int_\Omega (\mathcal{J}_h^t f)(y)\,dy + \mathcal{O}(h^{t-2}). \tag{4.6}$$

It therefore remains to show that the first term on the right hand side of (4.6) has the form $\sum_{ij}\lambda_{ij}q(x_{ij}-\cdot)$ with $\|\lambda\|_{p'} \leq K_h$ as in the statement of the Theorem.

In view of (4.4), the term in question is

$$\begin{aligned}\int_\Omega (\mathcal{J}_h^t f)(y)\,dy &= \sum_{rs}\int_{T_{rs}^+}(\mathcal{J}_h^t f)(y)\,dy + \sum_{rs}\int_{T_{rs}^-}(\mathcal{J}_h^t f)(y)\,dy \\ &= \sum_{rsi} f(x_{r\pm\alpha_i s\pm\beta_i})\int_{T_{rs}^\pm} p_{rsi}(y)\,dy =: \sum_{rsi} f(x_{r\pm\alpha_i s\pm\beta_i})b_{rsi}^\pm h^2,\end{aligned} \tag{4.7}$$

with $|b^{\pm}_{rsi}| \leq C_t$, and the summation is over the $(r,s) \in \Gamma^t_h$ and $i = 1, \ldots, t(t+1)/2$. Rearranging the order of summation by setting $k = r \pm \alpha_i$, $\ell = s \pm \beta_i$ shows that the last term in (4.7) equals

$$\sum_{k,\ell=0}^{N-1} c_{k\ell}\, h^2 f(x_{k\ell}) = \sum_{k,\ell=0}^{N-1} c_{k\ell}\, h^2\, w(x_{k\ell})\, q(x_{k\ell} - x), \tag{4.8}$$

where the coefficients $c_{k\ell}$ are obtained as

$$c_{k\ell} = \sum_{r\pm\alpha_i=k,\; s\pm\beta_i=\ell} b^{\pm}_{rsi}.$$

We deduce that $|c_{k\ell}| \leq t(t+1)C_t =: \frac{1}{2}c_t$, which in fact only depends on t. Setting $\lambda_{k\ell} = c_{k\ell}h^2 w(x_{k\ell})$ then proves the desired $\|\lambda\|_{p'} \leq K_h$:

$$\sum_{ij} \lambda_{ij}^{p'} \leq (c_t/2)^{p'} h^{2p'-2} \sum_{ij} h^2 w(x_{ij})^{p'} \leq (c_t/2)^{p'} h^{2p'-2} (\|w\|_{p'}^{p'} + \mathcal{O}(h)) \leq c_t^{p'} h^{2p'-2} \|w\|_{p'}^{p'}$$

for h sufficiently small. This proves the result. □

Corollary 4.2 *Let $\bar{u} > 0$ be the true image, and suppose $\log \bar{u} \in H^t(\Omega)$ for some $t \geq 3$ and that $\log \bar{u}$ may be represented as $q^+ * w$ for some $w \in L^{p'}(\Omega)$. Then, for any array v_h of data satisfying $\|v_h - \bar{v}_h\|_p \leq h^{-1}\epsilon_h$, the solution $\bar{u}_h$ of (P_1) satisfies the L^1-norm estimate*

$$\|\bar{u} - \bar{u}_h\|_1^2 = \mathcal{O}(h^{2t-4}) + \mathcal{O}(\epsilon_h h^{2/p-1}). \tag{4.9}$$

In particular, $\bar{u}_h$ converges to $\bar{u}$ in L^1-norm provided that $\epsilon_h h^{2/p-1} \to 0$ as $h \to 0$.

Proof. With the data satisfying $\|\bar{v}_h - v_h\|_p \leq h^{-1}\epsilon_h$, the constraint qualification (CQ) for program (P_1) is met, the optimal solution $\bar{u}_h$ exists and satisfies the dual relationship. Hence estimate (3.7) is satisfied.

As a consequence of our assumptions, Theorem 4.1 now applies to $u = 1 + \log \bar{u}$ and yields the estimate

$$E_h = E_h(1 + \log \bar{u}, q, K_h) = \mathcal{O}(h^{t-2}), \tag{4.10}$$

if K_h is chosen of the form $K_h = \mathcal{O}(h^{2/p})$ as specified in the statement of Theorem 4.1. Then (3.7) gives the claimed estimate

$$\|\bar{u} - \bar{u}_h\|_1^2 = \mathcal{O}(h^{2t-4}) + \mathcal{O}(\epsilon_h h^{2/p-1}).$$

This proves the result. □

We conclude this section with a brief discussion of the above hypothesis that $\log \bar{u} = q^+ * w$ for some $w \in L^{p'}(\Omega)$. It turns out that this is by no means an artificial requirement:

Proposition 4.3 *Suppose $q \in \mathcal{C}(\Omega)$, and let $\bar{u} \in \mathcal{C}(\Omega)$ be the true image satisfying $\bar{u} > 0$ and $\int_\Omega \bar{u}\,dx = 1$. Then there exists a unique function $\tilde{u} \in \mathcal{C}(\Omega)$ satisfying $\tilde{u} > 0$ and $\int_\Omega \tilde{u}\,dx = 1$ such that $q * \bar{u} = q * \tilde{u}$ and, in addition, $\log \tilde{u} = q^+ * w$ for some $w \in \mathcal{C}(\Omega)$. Moreover, if $\log \bar{u} \in C^k$, then also $\log \tilde{u} \in C^k$. In particular, if the kernel q is injective, $\log \bar{u} = q^+ * w$ for some $w \in \mathcal{C}(\Omega)$.*

Proof. According to Fredholm's alternative, the space $W = \{w \in \mathcal{C}(\Omega) : q * w = 0\}$ of solutions of the homogeneous equation is finite dimensional and spanned say by $w_1, \dots, w_r$. Moreover, the functions $f^+ \in \mathcal{C}(\Omega)$ of the form $f^+ = q^+ * w^+$ for some $w^+ \in \mathcal{C}(\Omega)$ are characterized by

$$\langle f^+, w_i \rangle = 0 \quad \text{for all } i = 1, \dots, r. \tag{4.11}$$

Now let us consider the following finite dimensional optimization program

$$(P) \quad \begin{array}{ll} \text{minimize} & \mathcal{I}(u) = \displaystyle\int_\Omega u(x) \log u(x)\,dx \\ \text{subject to} & u \in \bar{u} + W,\ u \geq 0, \\ & \displaystyle\int_\Omega u(x)\,dx = 1. \end{array}$$

Then (P) is feasible since $\bar{u}$ is admitted, and hence a unique solution $\tilde{u}$ exists. This follows in fact from the weak compactness of the level sets of $\mathcal{I}$ in $L^1(\Omega)$, and the strict convexity of $\mathcal{I}$, both proved in [4].

We show that $\tilde{u}$ is as claimed. Indeed, the Kuhn-Tucker conditions for (P) are the following:

1. $\nabla \mathcal{I}(\tilde{u}) + \mu \in W^\perp$;
2. $\tilde{u} \in \bar{u} + W$;
3. $\tilde{u} \geq 0,\ \displaystyle\int_\Omega \tilde{u}\,dx = 1$.

Since $\langle \nabla \mathcal{I}(\tilde{u}), h \rangle = \langle 1 + \log \tilde{u}, h \rangle$, condition 1. gives (4.11) for the function $f^+ = \log \tilde{u} + \mu + 1$, which means that $\log \tilde{u} + \mu + 1 = q^+ * w^+$ for some $w^+ \in \mathcal{C}(\Omega)$. Since $q^+ * 1$ is a constant function, the latter is equivalent to $\log \tilde{u} = q^+ * v^+$ for some $v^+ \in \mathcal{C}(\Omega)$, as desired.

Concerning the uniqueness of $\tilde{u}$, observe that any other function of this type would satisfy the Kuhn-Tucker conditions 1. - 3. above. However, by the convexity of (P), these characterize the optimal solutions of (P), and by the strict convexity of $\mathcal{I}$ (cf. [4]), the solution of (P) is unique. This completes the argument. □

Remark. Proposition 4.3 leads to the following slight improvement of Corollary 4.2:

Suppose $\log \bar{u} \in H^t(\Omega)$ *for some* $t > 2$. *Let* $\bar{v}_h = (q * \bar{u})_h$, *and suppose the data* v_h *for* (P_1) *satisfy* $\|v_h - \bar{v}_h\|_p \leq h^{-1}\epsilon_h$. *Then the solution* $\bar{u}_h$ *of* (P_1) *converges to* $\tilde{u}$ *in* L^1*-norm provided that* $\epsilon_h h^{2/p-1} \to 0$, *where* $\tilde{u}$ *is the function guaranteed by Proposition 4.3.*

This is the best we can hope for, since $q * \bar{u} = q * \tilde{u}$ by construction, and so $\bar{u}$ and $\tilde{u}$ could not possible be distinguished through any data sampled on the basis of model (1.2).

5 Stochastic Convergence

In this Section we proceed towards a stochastic convergence result for the model (P_2), which combines the optimization approach (P_1) with a priori smoothing of the data. To keep things easier, we will limit our considerations to smoothing operators of convolutional type. More precisely, we consider discrete kernel operators k which apply to a data array $v_h \in \mathbb{R}^{N\times N}$ on Ω_h via the formula:

$$(k * v_h)_{ij} = \sum_{r,s=0}^{N-1} k_{rs} v_{i-r,j-s}, \qquad 0 \leq i,j \leq N-1, \tag{5.1}$$

given an array of weights $k_{rs} \geq 0$, $0 \leq r,s \leq N-1$, satisfying $\sum_{rs} k_{rs} = 1$.

For a white noise sequence e_{ij} with mean zero and variance σ_e^2, the smoothed sequence $\tilde{e}_{ij} = (k * e)_{ij}$ has variance $\sigma_{\tilde{e}}^2 = (\sum_{rs} k_{rs}^2)\sigma_e^2$, so the noise variance is reduced by the smoothing factor $\rho(k)^2$, where $\rho(k) := (\sum_{rs} k_{rs}^2)^{1/2} < 1$.

In tandem with the smoothing factor $\rho(k)$, we shall need another characteristic $\theta(k)$ associated with a kernel operator of the above type, which is defined as

$$\theta(k) = \sum_{ij} k_{ij}\sqrt{i^2 + j^2}.$$

In the following, let us fix a sequence ϵ_h, and a sequence k_h of kernels with smoothing factor $\rho_h = \rho(k_h)$, and characteristic $\theta_h = \theta(k_h)$, and consider the following image restoration process defined for every $h > 0$:

Given the dirty image v_h sampled at the grid points of Ω_h, and according to the linear model (1.2), form the smoothed data $\tilde{v}_h = k_h * v_h$, and then calculate the unique solution $\bar{u}_h$ of the nonlinear maximum entropy estimation program

$$(P_2) \quad \begin{array}{ll} \text{minimize} & \mathcal{I}(u) = \int_\Omega u(x)\log u(x)\,dx \\ \text{subject to} & \|(q * u)_h - \tilde{v}_h\|_2 \leq h^{-1}\epsilon_h \\ & u(x) \geq 0, \ \int_\Omega u(x)\,dx = 1. \end{array}$$

The following result, involving conditions on ϵ_h, ρ_h and θ_h, gives sufficient conditions under which the error $\|\bar{u} - \bar{u}_h\|_1$ for the solution $\bar{u}_h$ of (P_2) converges to 0 with probability 1.

Theorem 5.1 *Let the true image $\bar{u}$ be strictly positive, and suppose for some $t \geq 3$, $\log \bar{u} \in H^t(\Omega)$, and that $\log \bar{u}$ is of the form $q^+ * w$ for some $w \in L^2(\Omega)$. Suppose that the smoothing factors ρ_h, the characteristics θ_h associated with the kernel operators k_h, and the tolerances ϵ_h satisfy the following set of conditions:*

$$(i) \sum_{h=1/(N-1)} \frac{\rho_h^2}{\epsilon_h^2} < \infty \qquad (ii) \quad \frac{h\theta_h}{\epsilon_h} \to 0 \qquad (iii) \quad \epsilon_h \to 0. \tag{5.2}$$

Then the L^1-error $\|\bar{u} - \bar{u}_h\|_1$ converges to zero with probability one.

Proof. The deterministic result Corollary 4.2 (applied in the case $p = 2$ and with v_h replaced by $\tilde{v}_h$) gives rise to the estimate

$$\|\bar{u} - \bar{u}_h\|_1^2 = \mathcal{O}(h^{2t-4}) + \mathcal{O}(\epsilon_h), \tag{5.3}$$

for the solution $\bar{u}_h$ of (P_2), provided that the data vector v_h is such that the smoothed data $\tilde{v}_h = k_h * v_h$ satisfy $\|\tilde{v}_h - \bar{v}_h\|_2 \leq h^{-1}\epsilon_h$ eventually. Since $t > 2$ and $\epsilon_h \to 0$, this shows $\|\bar{u}_h - \bar{u}\|_1 \to 0$ for such v_h. It therefore remains to show that $\|\tilde{v}_h - \bar{v}_h\|_2 \leq h^{-1}\epsilon_h$ eventually is satisfied with probability one.

Now observe that for every $h > 0$, and with L denoting the global Lipschitz constant of $\bar{v} = q * \bar{u}$, we have

$$P\{\|k_h * v_h - \bar{v}_h\|_2 > h^{-1}\epsilon_h\} \leq P\{\|k_h * v_h - k_h * \bar{v}_h\|_2 > h^{-1}\epsilon_h - L\theta_h\},$$

since

$$\{\omega : \|k_h * v_h(\omega) - \bar{v}_h\|_2 > h^{-1}\epsilon_h\} \subset \{\omega : \|k_h * v_h(\omega) - k_h * \bar{v}_h\|_2 > h^{-1}\epsilon_h - L\theta_h\}.$$

Indeed, for ω in the left hand set, we have

$$h^{-1}\epsilon_h < \|k_h * v_h(\omega) - \bar{v}_h\|_2 \leq \|k_h * v_h(\omega) - k_h * \bar{v}_h\|_2 + \|\bar{v}_h - k_h * \bar{v}_h\|_2, \tag{5.4}$$

and the second (deterministic) term in (5.4) may be estimated as follows:

$$\|\bar{v}_h - k_h * \bar{v}_h\|_2^2 = \sum_{ij} \Big(\sum_{rs} (k_h)_{rs} (\bar{v}_{i-r,j-s} - \bar{v}_{ij}) \Big)^2 \leq \sum_{ij} \Big(L \sum_{rs} (k_h)_{rs} \frac{\sqrt{r^2+s^2}}{N} \Big)^2 = L^2\theta_h^2.$$

Applying Chebysheff's inequality now gives the estimate

$$P\{\|k_h * (v_h - \bar{v}_h)\|_2 > h^{-1}\epsilon_h - L\theta_h\} \leq \frac{1}{(h^{-1}\epsilon_h - L\theta_h)^2} E(\|k_h * (v_h - \bar{v}_h)\|_2^2) =: A^2 \quad (5.5)$$

and according to the model (1.2), the $E(\cdot)$ term in (5.5) is

$$E(\|k_h * e_h\|_2^2) = \sum_{ij} E(\tilde{e}_{ij}^2) = \sum_{ij}\sum_{rs}\sum_{r's'}(k_h)_{rs}(k_h)_{r's'}E(e_{i-r,j-s}e_{i-r',j-s'}) = N^2\rho_h^2\sigma_e^2,$$

the latter using the fact that the e_{ij} are white noise with mean zero and variance σ_e^2. Consequently, the last term in (5.5) is

$$A^2 = \frac{N^2\rho_h^2\sigma_e^2}{(h^{-1}\epsilon_h - L\theta_h)^2} \leq K\left(\frac{\rho_h}{\epsilon_h}\right)^2,$$

where we use $N = \mathcal{O}(h^{-1})$ and $(h^{-1}\epsilon_h - L\theta_h)/(h^{-1}\epsilon_h) \to 1$, which follows from (ii).

Let $F = \{\omega : \|\tilde{v}_h(\omega) - \bar{v}_h\|_2 \leq h^{-1}\epsilon_h \text{ eventually}\}$. Then, for every $\delta > 0$, we have

$$\begin{aligned} P(F) &= 1 - P\{\forall h \leq \delta\ \exists h' \leq h\ \|\tilde{v}_{h'} - \bar{v}_{h'}\|_2 > h'^{-1}\epsilon_{h'}\} \\ &\geq 1 - P\{\exists h \leq \delta\ \|\tilde{v}_h - \bar{v}_h\|_2 > h^{-1}\epsilon_h\} \\ &\geq 1 - \sum_{h\leq\delta} P\{\|\tilde{v}_h - \bar{v}_h\|_2 > h^{-1}\epsilon_h\} \geq 1 - K\sum_{h\leq\delta}\left(\frac{\rho_h}{\epsilon_h}\right)^2. \end{aligned}$$

Since $\delta > 0$ is arbitrary, the last term may be made arbitrarily close to 1 regarding the fact that by assumption (i) the series $\sum_h \rho_h/\epsilon_h$ converges quadratically. This shows $P(F) = 1$ and completes the argument. □

Corollary 5.2 *With the same basic assumptions as in Theorem 5.1, suppose the following set of conditions is satisfied:*

$$(i)\quad \frac{\rho_h}{\epsilon_h} \to 0 \quad (ii)\quad \frac{h\theta_h}{\epsilon_h} \to 0 \qquad (iii)\quad \epsilon_h \to 0 \qquad (5.6)$$

Then the L^1-error tends to 0 in probability, i.e., $\|\bar{u} - \bar{u}_h\|_1 \xrightarrow{\mathcal{P}} 0$.

Proof. For fixed $\alpha > 0$ we have to show $P\{\|\bar{u} - \bar{u}_h\|_1 \geq \alpha\} \to 0$. By the deterministic result and condition (iii), $\|\tilde{v}_h - \bar{v}_h\|_2 \leq h^{-1}\epsilon_h$ implies $\|\bar{u} - \bar{u}_h\|_1^2 \leq K(h^{2t-4} + \epsilon_h) < \alpha^2$ for some $K > 0$ and $h < h_0$, say. Therefore

$$\{\omega : \|\bar{u} - \bar{u}_h(\omega)\|_1^2 \geq \alpha^2\} \subset \{\omega : \|\tilde{v}_h(\omega) - \bar{v}_h\|_2 > h^{-1}\epsilon_h\}$$

for all $h < h_0$. It then suffices to show that $P\{\|\tilde{v}_h - \bar{v}_h\|_2 > h^{-1}\epsilon_h\} \to 0$, and this is precisely the argument given in the proof of Theorem 5.1, where condition (5.6) (i) is now used instead of (5.2) (i). This completes the argument. □

Remark 1. Naturally, we have to show that the set of conditions (5.2) resp. (5.6) is consistent. We shall do this in particular for the d-nearest neighbour type kernel estimators. Let d_h be a sequence of integers, and define the kernels k_h through the array of weights

$$(k_h)_{rs} = \begin{cases} \dfrac{1}{(2d_h+1)^2} & \text{for } |r| \le d_h \text{ and } |s| \le d_h, \\ 0 & \text{else.} \end{cases}$$

Then we easily find the smoothing factor and the characteristic to be

$$\rho_h^2 = \sum_{rs} (k_h)_{rs}^2 = (2d_h+1)^{-2} = \mathcal{O}(d_h^{-2})$$

resp.

$$\theta_h = \Big(\sum_{|r|,|s| \le d_h} \sqrt{r^2+s^2} \Big) \Big/ (2d_h+1)^2 = \mathcal{O}(d_h).$$

The conditions (5.2) therefore read as:

$$(i) \quad \sum_{h=1/(N-1)} (d_h\epsilon_h)^{-2} < \infty \quad (ii) \quad hd_h/\epsilon_h \to 0 \qquad (iii) \quad \epsilon_h \to 0. \tag{5.7}$$

But these may be arranged e.g. with $h = \mathcal{O}(1/N)$, $h^{-1}\epsilon_h = N^{4/5}$, $d_h = \mathcal{O}(N^{3/4})$, for then $d_h \to \infty$, $hd_h/\epsilon_h = \mathcal{O}(N^{-1/20}) \to 0$, $\epsilon_h = N^{-1/5} \to 0$, and $\sum (d_h\epsilon_h)^{-2} \doteq \sum N^{-11/10} < \infty$.

Remark 2. In Section 1, the choice $\epsilon_h = \mathcal{O}(\rho_h)$ was suggested, which in the situation of Remark 1 above gives $\epsilon_h = \mathcal{O}(d_h^{-1})$. But then it follows that condition (i) is violated, for $d_h\epsilon_h = \mathcal{O}(\rho_h d_h) = \mathcal{O}(1)$ does not tend to 0. Our method of proof in fact suggests that a slightly larger tolerance $\epsilon_h = \rho_h\sigma_e/\tau_h$, where $\tau_h \to 0$, is correct. More precisely, by Theorem 5.1 and Remark 1 above, the set of conditions

$$(i) \quad \sum \tau_h^2 < \infty \quad (ii) \quad d_h^2 h\tau_h \to 0 \quad (iii) \quad \frac{1}{d_h\tau_h} \to 0 \tag{5.8}$$

guarantees L^1 convergence a.e., while the weaker $(i')\ \tau_h \to 0$ still gives L^1-convergence in probability (Corollary 5.2).

Remark 3. The set of conditions (5.8) may be satisfied e.g. by choosing $\tau_h = h^\alpha$, $d_h = \mathcal{O}(h^{-\beta})$ with $1 < 2\alpha < 2\beta < 1+\alpha$, while for (i') $0 < 2\alpha < 2\beta < 1+\alpha$ is sufficient. In particular, this shows $d_h \ll N$ and also $\tau_h^{-1} \ll N$, $\epsilon_h = N^\alpha\rho_h\sigma_e$. Observe in particular that for convergence in probability, the null sequence τ_h may be chosen arbitrarily slow, so that the choice $\epsilon_h = \rho_h\sigma_e$ suggested in Section 1 might still be acceptable in practice (see Section 7).

6 Practical Aspects

We briefly discuss the computational aspects of the model (P_1) resp. (P_2) (in the case $p = 2$). Notice that alternative approaches have been presented e.g. by Frieden [14], Burch et al. [11], and Haralick et al. [15], Andrews et al. [3]. Perhaps closest to our present model (P_1) is Auyeung et al. [3], where the performance index is the Burg entropy measure, but the size of the images used in that reference is too small to allow for a realistic comparison. For a comparison of the maximum entropy based methods with other techniques see in particular Trussell [22].

While the papers [14], [11] and [15] propose different strategies, we claim that the most natural approach to solving program (P_1) is via its dual (D_1), which is a unconstrained optimization program and may therefore be solved using methods designed for unconstrained problems. As we have reported in [19], Newton's method did not lead to a satisfactory results here, but an adaption of the conjugate gradient method to maximizing the non-quadratic objective $\mathcal{J}(\lambda, \mu)$ as proposed by Fletcher and Reeves exhibits quite satisfactory performance. Depending on the size and the symmetries of the mask q and the tolerance level ϵ, a restoration of a degraded 200×320 clown image, as used for our experiments, takes between 4 and 10 minutes CPU.

An alternative approach in the spirit of [14] is to replace the tolerance type model (P_1) with a penalty approach

$$(\tilde{P}_1) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\int_\Omega u(x) \log u(x)\, dx + \frac{1}{2} C_h h^2 \|Q_h u - v_h\|_2^2 \\ \text{subject to} & u(x) \geq 0, \ \displaystyle\int_\Omega u(x)\, dx = 1. \end{array}$$

Here the associated dual $(\tilde{D}_1)$, related to $(\tilde{P}_1)$ via the same return formula (2.1), is

$$(\tilde{D}_1) \qquad \begin{array}{ll} \text{maximize} & \mathcal{J}(\lambda, \mu) = \displaystyle\sum_{ij} \lambda_{ij} v_{ij} + \mu - \int_\Omega \exp\{Q_h^* \lambda(x) + \mu - 1\}\, dx - \frac{1}{2 C_h h^2} \|\lambda\|_2^2 \\ \text{subject to} & \lambda \in \mathbb{R}^{N \times N}, \ \mu \in \mathbb{R}. \end{array}$$

This resembles the dual (D_1), but the nonsmooth term $h^{-1} \epsilon_h \|\lambda\|_2$ ranging in (D_1) has now been replaced with the more convenient term $\frac{1}{2C_h h^2}\|\lambda\|_2^2$. Surprisingly, the models (P_1) and $(\tilde{P}_1)$ are equivalent in a sense we are going to specify.

Notice that, by convexity, the optimal solution $\bar{u}_h$ of (P_1) satisfies the equality $\|Q_h u - v_h\|_2 = h^{-1}\epsilon_h$, i.e. the inequality constraint will be active – unless the global minimum $u_{\min}$ of the Boltzmann-Shannon objective $\mathcal{I}$ lies inside the convex set $\|Q_h u - v_h\|_2 \leq h^{-1}\epsilon_h$, which depending on q is an ellipsoid or an elliptic cylinder. Since $u_{\min}$ is a constant function, this is never the case for any problem of practical relevance (cf. [19]). Disregarding this case, we find that the optimal solution $\bar{u}_h$ of (P_1) satisfies the following necessary optimality conditions: There exist multipliers $\rho \geq 0$ and μ satisfying

1. $\nabla\mathcal{I}(u) + \rho\, Q_h^*(Q_h u - v_h) + \mu 1 = 0$;
2. $\displaystyle\int_\Omega u(x)\,dx = 1$;
3. $\|Q_h u - v_h\|_2 = h^{-1}\epsilon_h$.

On the other hand, the necessary optimality conditions for program $(\tilde{P}_1)$ are as follows: There exists a multiplier μ such that

1. $\nabla\mathcal{I}(u) + C_h h^2\, Q_h^*(Q_h u - v_h) + \mu 1 = 0$;
2. $\displaystyle\int_\Omega u(x)\,dx = 1$.

This means that the optimal solution $\bar{u}_h$ of $(\tilde{P}_1)$ is as well optimal for (P_1) if we let $\rho := C_h h^2$ and $\epsilon_h := h\,\|Q_h\bar{u}_h - v_h\|_2$, and conversely, by the uniqueness of the solutions of both programs, the solution $\bar{u}_h$ of (P_1) solves $(\tilde{P}_1)$ with the choice $C_h = h^{-2}\rho$. Therefore, for the numerics, instead of solving (D_1), we may as well use the dual program $(\tilde{D}_1)$ in tandem with the return formula (2.1). While the dual objective of $(\tilde{D}_1)$ is smooth and therefore somewhat better suited than the objective of (D_1), the extra work required for solving via $(\tilde{D}_1)$ is to provide the correct choice of the penalty constant C_h to match say a prescribed tolerance level $\|Q_h\bar{u}_h - v_h\|_2 = h^{-1}\epsilon_h$ (as derived in earlier sections). This may be obtained by performing a line search in C_h, which in practice usually requires a very limited number of steps.

Let us show that the duality between $(P_1), (D_1)$ and $(\tilde{P}_1), (\tilde{D}_1)$ could be used to estimate the size of the penalty constants C_h as compared to the corresponding tolerance level ϵ_h. Writing down the equation $V(P_1) - V(\tilde{P}_1) = V(D_1) - V(\tilde{D}_1)$ gives

$$\frac{1}{2C_h h^2}\|\lambda_h\|_2^2 - h^{-1}\epsilon_h\|\lambda_h\|_2 = \frac{C_h}{2}\epsilon_h^2. \tag{6.1}$$

Here we use the fact that the return formula (2.1) is the same for both dual pairs, so if $\bar{u}_h$ is the solution for both (P_1) and $(\tilde{P}_1)$, the same must be the case for the duals: (λ_h, μ_h) solves (D_1) and $(\tilde{D}_1)$. But now the quadratic equation (6.1) has the double solution

$$C_h = \frac{\|\lambda_h\|_2}{h\epsilon_h}, \tag{6.2}$$

which could be considered as explicit if we knew the dual optimal solution λ_h or at least could find some means to estimate its norm. Nevertheless, since $\bar{u}_h \to \bar{u}$, and using $\|\lambda_h\|_2 = \mathcal{O}(h)$ (cf. Corollary 4.2), (6.2) at least tells us that $C_h = \mathcal{O}(\epsilon_h^{-1}) \to \infty$.

Another practical aspect of the dual models (D_1) resp. $(\tilde{D}_1)$ is that the optimization be best split in two stages, a line search in μ, and a maximization over λ:

$$\max_{\mu\in\mathbb{R}}\ \max_{\lambda\in\mathbb{R}^{N\times N}} \mathcal{J}(\lambda,\mu)$$

Indeed, as has been observed by various authors working with the noise model (1.3), the constraint $\int u(x)\,dx = 1$ may in practice often be ignored, so the line search over μ will either become completely superfluous or at least require but a small number of steps. (It should be said, however, that the constraint could not be omitted in all cases. For instance, among the model masks used for the present experiments (see also [19]), we found that e.g. the pillbox ring masks have a tendency to underscore the total energy balance).

Concerning the model (P_2), with previously smoothed data, let us mention that in practice, instead of applying a linear kernel operator, one would probably prefer more robust statistics like a nonlinear median filter in order to obtain a satisfactory noise reduction. The reason why our present theoretical investigations were based on linear kernel operators is primarily due to the fact that they are easier to analyze. The model (P_2) could in fact be considered with any type of smoothing device, and one would then only have to estimate the second (deterministic) term occurring in the inequality (5.4) in order to obtain a convergence result in the spirit of Theorem 5.1.

7 Experiments

We used the 200×320 clown image displayed in Figure 1 for our simulations. In Figure 2, the clown was blurred using a 7×7-supported mask q of entries $1/49$, and normal white noise with variance $\sigma_e^2 = 15.55$ was added. This lead to a global signal-to-noise ratio of $S/N = 15.99\,\mathrm{dB}$, (cf. [19] or [20] for this logarithmic scale).

Figures 3 to 7 present various restorations based on the models (P_1) and (P_2). Notice here that for a rectangular $N \times M$ image, the tolerance in program (P_1) has to be chosen as $\sqrt{NM}\epsilon$. In Figure 3 we used the default value $\epsilon = 3.91$ as suggested by the law of large numbers (Section 1). In Figure 4, the extended model (P_2) was used. The blurred-and-noisy image was smoothed with the 3×3 supported mask k of entries $1/9$, introducing a smoothing factor of $\rho(k) = 1/3$. The tolerance level was consequently reduced to $\tilde{\epsilon} = \epsilon/3 = 1.32$. In Figure 5, according to the results obtained by our convergence analysis, a somewhat larger tolerance $\epsilon = 1.97$ was allowed after previously smoothing the degraded image with the same linear filter.

As can be seen, the restorations in Figures 4 and 5 suffers from ringing effects along the boundaries. In Figure 6, therefore, ringing was avoided by embedding the image into a somewhat larger frame filled up with zeros. As experiments show, this frame should have width at least the diameter of the mask q, and in our case was chosen of width 9. The same linear a priori smoothing was used, and the tolerance level was now $\epsilon = 2.11$.

Following a common suggestion in the digital image literature, in Figure 7, a priori smoothing was performed using a nonlinear 3×3-supported median filter. Framing to avoid wrapping around effects and the tolerance level $\epsilon = 2.11$ were used.

Finally, Figure 8 shows a restoration based on the linear restoration filter proposed by Hunt (cf. [16, 20, 1]), which arises from the same noise model using the performance

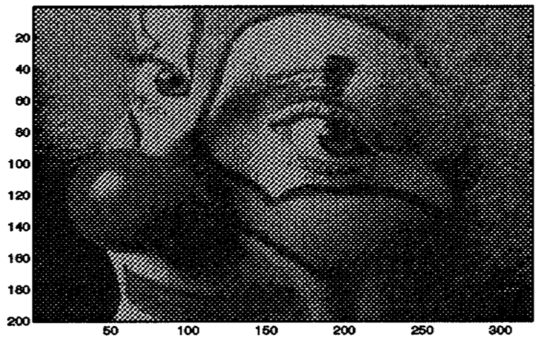

Figure 1. True 200 × 320 clown image

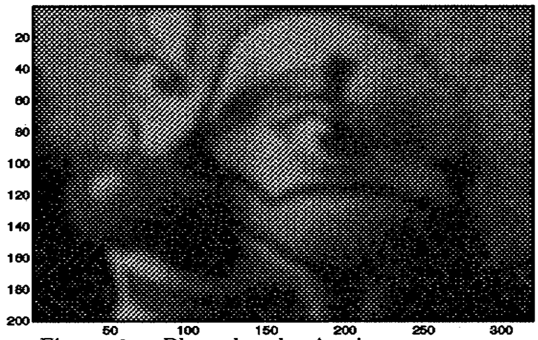

Figure 2. Blurred-and-noisy image; $S/N = 15.4\,dB$

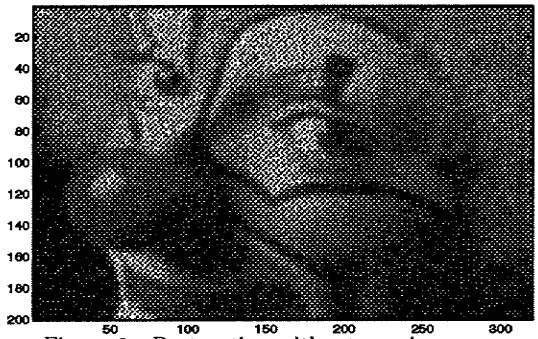

Figure 3. Restoration without previous linear smoothing $\epsilon = 3.94$

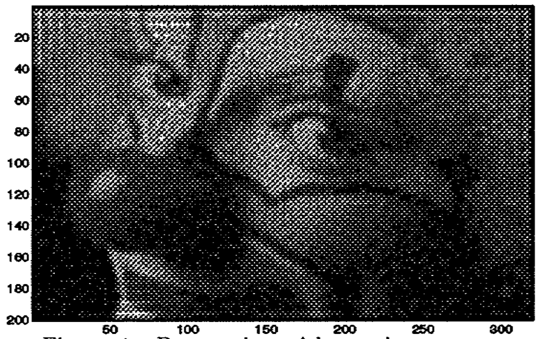

Figure 4. Restoration with previous linear smoothing, $\epsilon = 1.32$

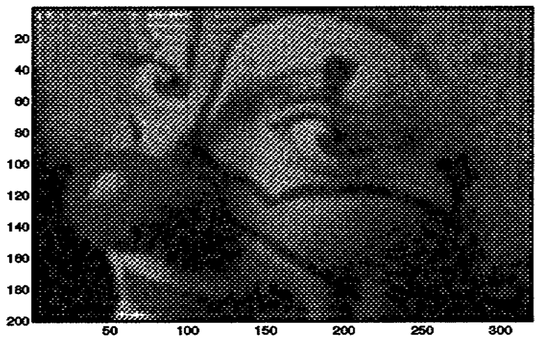

Figure 5. Restoration with previous linear smoothing; $\epsilon = 1.97$

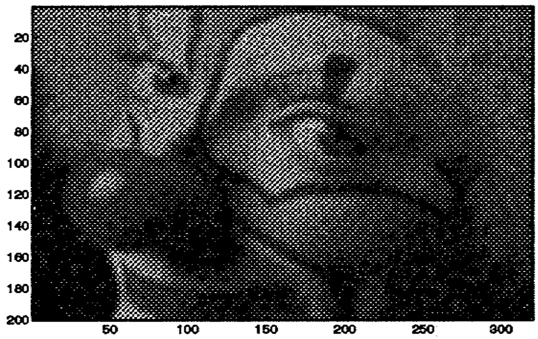

Figure 6. Restoration with linear smoothing and framing; $\epsilon = 2.11$

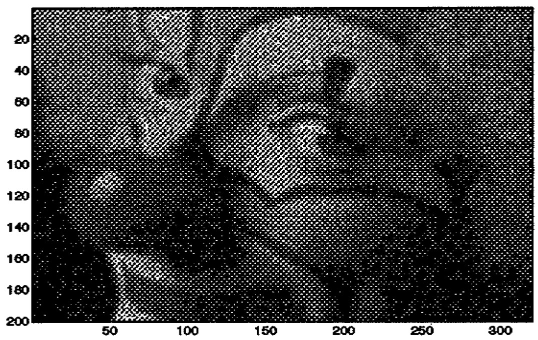

Figure 7. Restoration with previous nonlinear smoothing; $\epsilon = 2.11$

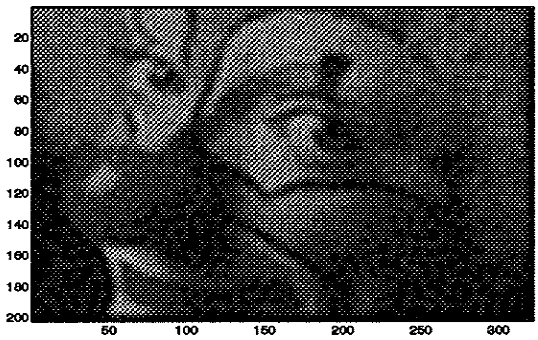

Figure 8. Restoration with linear filter

index $\mathcal{I}(u) = \int_\Omega |\Delta u|^2\,dx$. Reference [16] presents a clever way of solving the corresponding discretized Euler/Lagrange equation using the 2D Fast Fourier Transform (FFT2), which is considerably faster than the nonlinear approach based on (P_1).

Our experiments show that Hunt's approach when combined with previous smoothing of the degraded image is superior to a direct least squares inversion of the linear system $(q * u)_h = \tilde{v}_h$, which would be the classical approach to the inversion problem based on regression methods.

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