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Vol. 76, No. 2, February 1988
Printed in Belgium

Continuous Affine Support Mappings for Convex Operators

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Communicated by the Editors

Received November 15, 1986

We investigate the problem of existence of continuous affine resp. linear support mappings for convex resp. sublinear partially defined operators. We prove that a weakly Borel measurable convex operator defined on a closed convex subset of a locally convex Fréchet space with values in an ordered locally convex vector space with the least upper bound property and an order-unit inducing the topology in fact has continuous affine support mappings. We give an example indicating that the order-unit being present is essential. A step towards our result is provided by a variant of the sandwich theorem obtained for the case of weakly Borel measurable convex and concave operators. Finally, we obtain necessary and sufficient conditions for the existence of a continuous affine support mapping h for a convex operator φ such that $h(x_0) = \varphi(x_0)$ holds for a fixed point x_0 in terms of the upper directional derivate $\partial\varphi(x_0; \,)$ for φ at x_0.

INTRODUCTION

Let E be a vector space and let F be an ordered vector space with the least upper bound property. The Hahn–Banach theorem asserts that every sublinear operator $\varphi: E \to F$ has a linear support mapping $f: E \to F$ (i.e., a linear mapping f with $f(x) \leqslant \varphi(x)$ for all $x \in E$). In addition, if E and F are locally convex vector spaces and if the order on F is normal, then continuity of φ naturally implies the continuity of f. This situation thoroughly changes if the sublinear operator φ is only defined on a cone C in E. It is known that even in the case $F = \mathbb{R}$, such a partially defined sublinear operator need not have any linear support functional. Moreover, once the existence of a linear support mapping for a partially defined sublinear operator has been established, it is by no means clear whether the continuity of φ on C implies the continuity of f on the subspace $C - C$ generated by C.

The aim of this paper is to find conditions ensuring the existence of con-

0022-1236/88 $3.00

tinuous affine resp. linear support mappings for partially defined convex resp. sublinear operators. In Section 1 we ask for conditions under which linear support mappings for partially defined (not necessarily convex) operators are automatically continuous. We treat the case where φ is continuous on its domain C and the case where φ is weakly Borel measurable. For instance, we prove that if φ is a weakly Borel measurable operator defined on a closed generating cone C in some completely metrizable locally convex vector space E with values in a normally ordered locally convex vector space F, then every linear support mapping f for φ is necessarily continuous.

In Section 2 we obtain a generalization of the classical Hörmander theorem (see [Hö]). We prove that a partially defined sublinear operator φ with values in an ordered locally convex vector space F with the least upper bound property and an order-unit inducing the topology is the upper envelope of its continuous linear support mappings if and only if it is lower semi-continuous. We provide an example indicating that this result is no longer true if the existence of an order-unit in F is omitted.

In Section 3 we prove a sandwich theorem in the spirit of [Zo_2] but under somewhat different conditions. For a convex operator φ with domain X and a concave operator ψ with domain Y such that 0 is an algebraic interior point of $X - Y$ and with $\psi \leqslant \varphi$ on $X \cap Y$, we establish the existence of a continuous affine mapping h such that $h \geqslant \psi$ on Y, $h \leqslant \varphi$ on X under the assumption that F has the least upper bound property, φ, ψ, are weakly Borel measurable, X, Y are closed and E is a completely metrizable locally convex vector space.

In Section 4 we obtain two consequences of our sandwich theorem concerning the existence of continuous affine resp. linear support mappings for weakly Borel measurable partially defined convex resp. sublinear operators. Moreover, we obtain necessary and sufficient conditions for the existence of continuous affine support mappings h for convex operators φ satisfying $h(x_0) = \varphi(x_0)$ at a fixed point x_0 in terms of the directional derivate operator $\partial\varphi(x_0;\ )$ of φ at x_0. We conclude our paper with several limiting examples.

Notations

Throughout we assume that E is a real locally convex vector space and that F is a real ordered locally convex vector space with normal positive cone F_+, i.e., F has a base of neighborhoods of 0 consisting of order-convex sets. F is said to have the least upper bound property (l.u.b.p.) if every subset B of F which is order-bounded below has an infimum. An algebraic interior point of F_+ is called an order-unit for F. An order-unit e on F