Local convergence of the method of alternating projections

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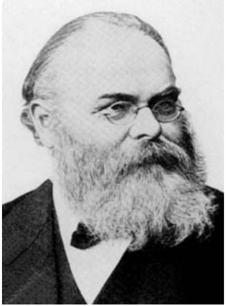
Alternating projections invented by Hermann Amandus Schwarz in 1869

H. A. Schwarz. Über einen Grenzübergang durch alternirendes Verfahren.

Vierteljahresschrift der Naturforschenden Gesellschaft in Zürich, 15(1870), pp. 272–286.



- Used to solve Dirichlet problem rigorously
- First domain decomposition method ever
- Modern re-interpretation by P.-L. Lions 1978, 1988-89.

Method erroneously attributed to J. von Neumann :

S.L. Sobolev. L'algorithme de Schwarz dans la théorie de l'élasticité. Comptes Rendus de l'Académie des Sciences de l'URSS, IV :243–246, 1936.

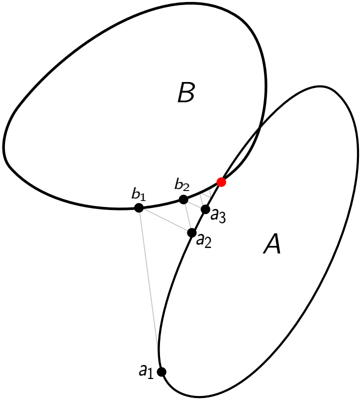
R. Courant, D. Hilbert. Verfahren der mathematischen Physik, Band 2 1930s.

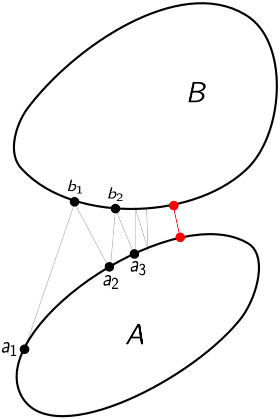
J. von Neumann. Functional Operators II. Lecture Notes 1950

- Presents no citations
- Claims that original version in 1933 has it already

Convex Alternating Projections

- Schwarz, Sobolev, v. Neumann : Subspaces
- L.M. Bregman : Weak convergence for convex sets in Hilbert space. 1965
- H.H. Bauschke : Convex case essentially settled 1993.
- H. Hundal. Norm convergence may fail, 2002.


B
b1
b2
a3
a2
A
a1


B
b1
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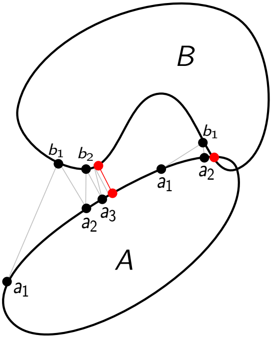
B
b_1
b_2
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a_1
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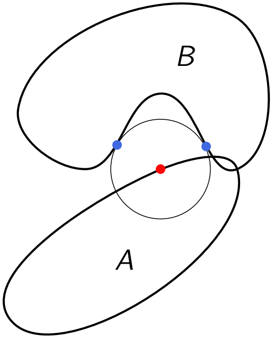
a_3
A
a_1


B
A

Given : closed sets A, B in $\mathbb{R}^n$

$$A \cap B \neq \emptyset$$

Want : solution x of feasibility problem

$$x \in A \cap B$$

Method :

$$b_1 \in P_B(a_1),\ a_2 \in P_A(b_1),\ b_2 \in P_B(a_2),\ a_3 \in P_A(b_2),\ \ldots$$

or

$$a_1 \xrightarrow{P_B} b_1 \xrightarrow{P_A} a_2 \xrightarrow{P_B} b_2 \xrightarrow{P_A} \ldots$$

Non-convex Alternating Projections :

- Are there applications ?
- Conditions for local convergence ?
- May convergence fail ?

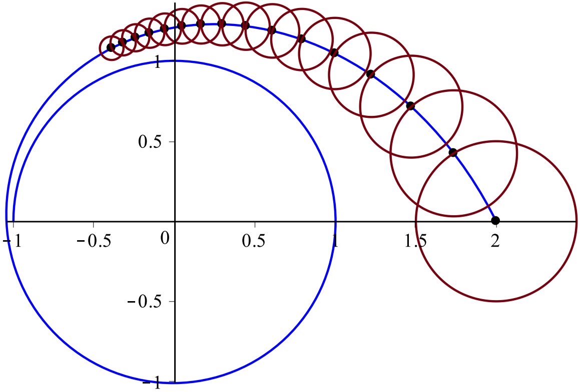
Failure of convergence

Theorem

(Combettes, Trussell 1990). *Let A, B be closed. Suppose the sequence of alternating projections a_k, b_k is bounded and satisfies $a_k - b_k \to 0$. Then the set of accumulation points of a_k, b_k is either singleton or a compact continuum. Every accumulation point is a solution of the feasibility problem.*

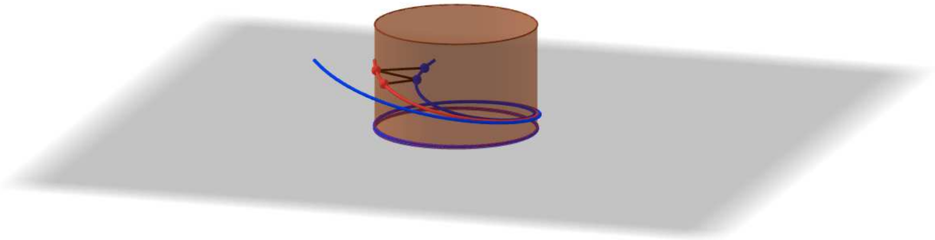
Theorem

(Bauschke, Noll 2013). *The case of a non-trivial compact continuum may indeed occur.*



$$A = \{1^{\text{st}}, 3^{\text{rd}}, 5^{\text{th}}, 7^{\text{th}}, \ldots\} \cup C, \quad B = \{2^{\text{nd}}, 4^{\text{th}}, 6^{\text{th}}, \ldots\} \cup C.$$

$$A \cap B = C = \{z : |z| = 1\}$$



$$A = \{(\cos t, \sin t, s) : 0 \leq s \leq 1, 0 \leq t \leq 2\pi\}$$
$$B = \{(\cos t(1 + e^{-t}), \sin t(1 + e^{-t}), e^{-2t}) : 0 \leq t \leq \infty\}$$

Bauschke, Noll (2014, Archiv der Mathematik)

Are there applications?

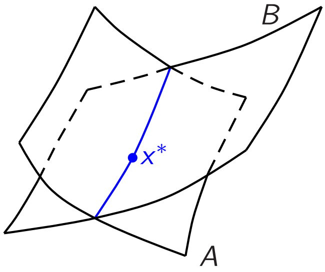
Non-convex Alternating Projections used in :

- Color plane interpolation (de-mosaicking)
- De-noising of time-series (Cadzow's basic algorithm, Singular Spectrum Analysis)
- Inverse eigenvalue problems
- Pole placement (control)
- Synthesis of low-order feedback controllers (control)
- Road profile design (western Canada)
- Recovery of lost image blocks in JPEG and MPEG images
- Sparse affine feasibility (for error correction in linear codes)
- Packings in Grassmannian manifolds (wireless communication)
- EM-algorithm for Gaussian laws
- Phase retrieval

Local convergence

Theorem

(A.S. Lewis, J. Malick 2008). *Let A, B be C^2-manifolds in $\mathbb{R}^n$ intersecting transversally at $x^* \in A \cap B$. Then there exists a neighborhood U of x^* such that every alternating sequence a_k, b_k which enters U converges to some $a^* \in A \cap B$ with R-linear speed.*



Transversality

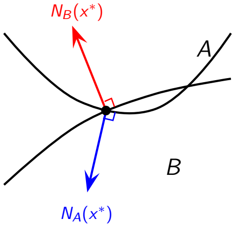
$$T_A(x^*) + T_B(x^*) = \mathbb{R}^n$$

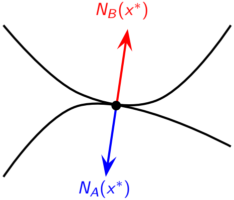
Theorem

(A.S. Lewis, R. Luke, J. Malick 2009). *Suppose*

1. *There exists $x^* \in A \cap B$ such that $N_A(x^*) \cap -N_B(x^*) = \{0\}$ (replaces transversality).*
2. *B is super-regular (replaces convexity).*

Then there exists a neighborhood U of x^* such that every alternating sequence a_k, b_k which enters U converges to some $a^* \in A \cap B$ with R-linear speed.

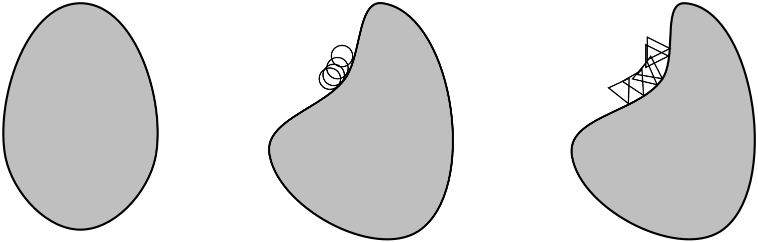




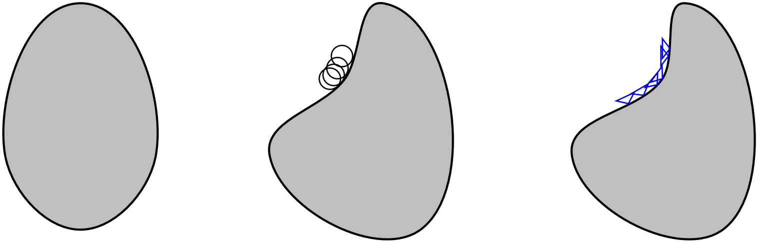
A.S. Lewis, J. Malick (2008). Alternating projections on manifolds.
Math. Oper. Res.

A.S. Lewis, R. Luke, J. Malick (2009). Local linear convergence for alternating and averaged non-convex projections.
Foundations Comp. Math.

- Transversality too restrictive. Two non-parallel lines in $\mathbb{R}^2$ intersect transversally, but no longer in $\mathbb{R}^3$
- Same for $N_A(x^*) \cap -N_B(x^*) \subset \{0\}$.
- Need an additional regularity hypothesis called super-regularity.

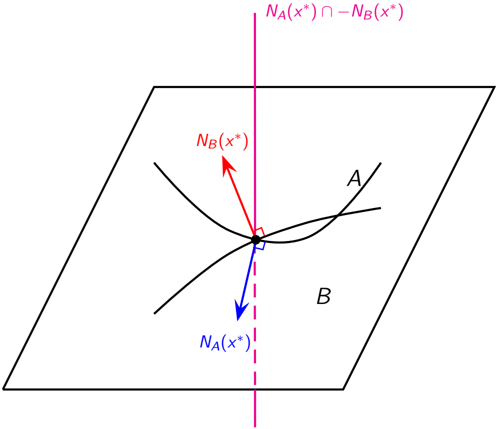


convex prox-regular super-regular



convex prox-regular super-regular

H.H. Bauschke, D.R. Luke, H.M. Phan, X. Wang (2013). Restricted normal cones and the method of alternating projections.
Set-valued Var. Anal.



Use restricted normal cones instead :

$$N_A^B(x^*) = \text{ normals to } A \text{ at } x^* \text{ pointing into } B$$

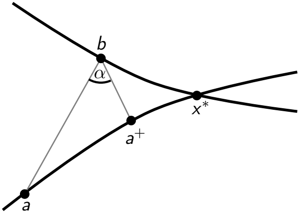
Transversality at x^* becomes :

$$N_A^B(x^*) \cap -N_B^A(x^*) \subset \{0\}$$

Works better, but still not good enough.

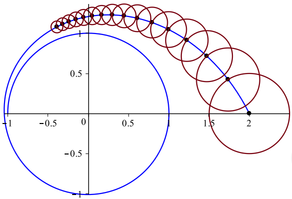
Definition

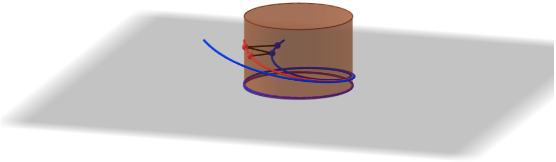
(Noll, Rondepierre 2013). Transversality is when α stays away from 0° in neighborhood of x^*.



What happens when the intersection is tangentiel ?

- Is failure of convergence due to the lack of regularity ?
- or is it because the intersection is (too) tangential ?





$\alpha \approx 180^\circ$ (transversal)

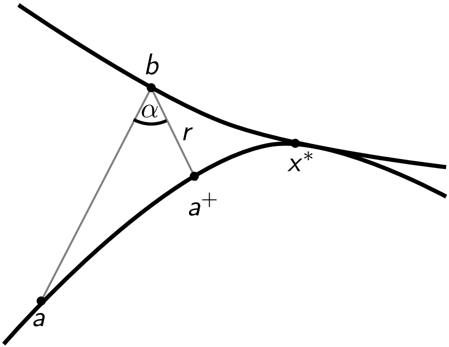
Regularity missing

intersection tangential

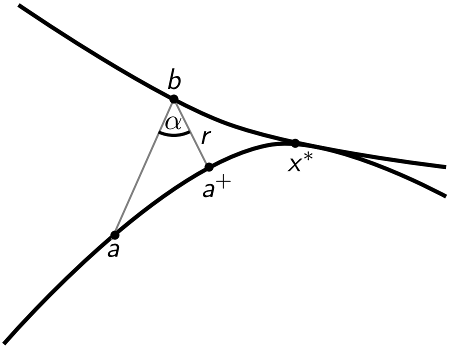
Regularity OK

How to deal with tangential intersection ?

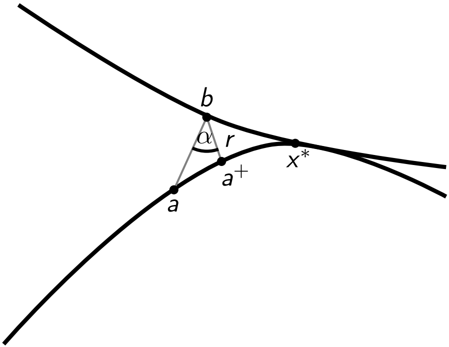
Noll, Rondepierre 2013 :



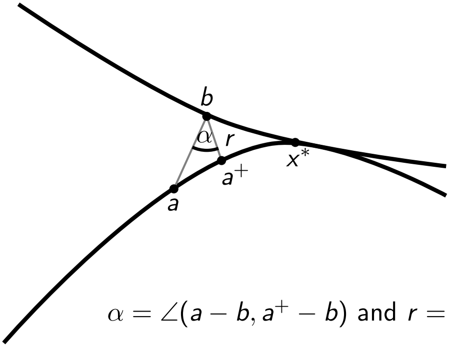
Tangential intersection :



Tangential intersection :



Tangential intersection :



$\alpha = \angle(a - b, a^+ - b)$ and $r = \|b - a^+\|$

both shrink to 0 as we approach x^*

Definition

(Noll, Rondepierre 2013). The sets A, B satisfy the angle condition at $x^* \in A \cap B$ if there exists $\gamma > 0$, $\omega \in [0, 2)$ and a neighborhood U of x^* such that for every building block $a \xrightarrow{P_B} b \xrightarrow{P_A} a^+$ in U we have

$$\frac{\sin^2 \alpha}{r^\omega} \geq \gamma$$

- Tangential intersection means α and r both shrink to 0.
- Angle condition means they shrink in controlled fashion. Angle does not shrink too fast.
- Special case $\omega = 0$ gives back transversality (angle does not shrink, but distance r does).

Theorem

(Noll, Rondepierre 2013). *Suppose there exists $x^* \in A \cap B$ such that*

1. *A, B satisfy the ω-angle condition at x^*.*
2. *B is $\omega/2$-Hölder regular at x^* with respect to A.*

Then there exists a neighborhood U of x^* such that every alternating sequence a_k, b_k which enters U converges to some point $a^* \in A \cap B$. The speed of convergence is

$$\|a_k - a^*\| = \mathcal{O}\left(k^{-\frac{2-\omega}{2\omega}}\right), \qquad \|b_k - a^*\| = \mathcal{O}\left(k^{-\frac{2-\omega}{2\omega}}\right)$$

Special case $\omega = 0$ gives R-linear convergence

Theorem

Noll, Rondepierre 2013). *Suppose A, B are sub-analytic sets and $x^* \in A \cap B$. Then there exists $\omega \in [0, 2)$ such that A, B intersect with ω-angle condition at x^*.*

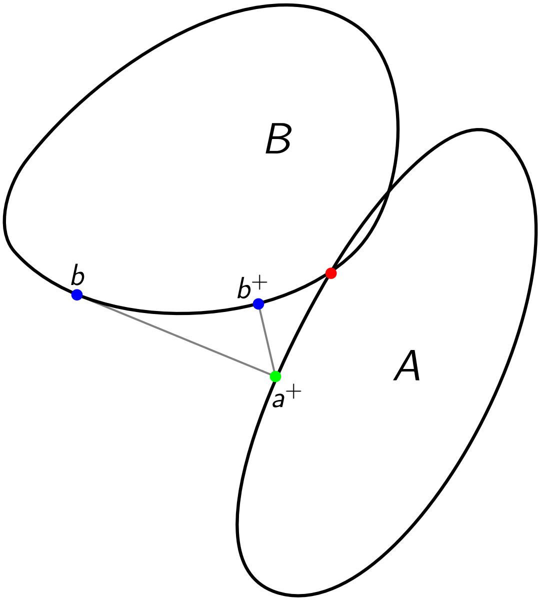
Semi-analytic set :

$$A = \bigcup_{i=1}^{N} \bigcap_{j=1}^{M} \{x \in \mathbb{R}^n : \phi_{ij}(x) = 0, \psi_{ij}(x) > 0\}$$

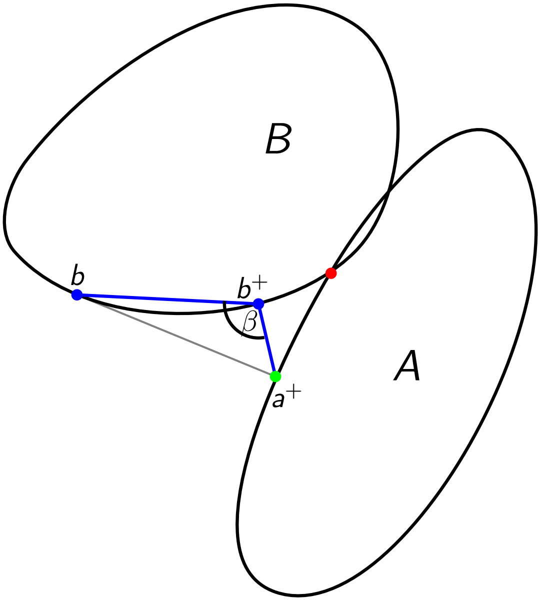
with real-analytic functions ϕ_{ij}, ψ_{ij}.

A sub-analytic $\iff$ $\forall a \in A \; \exists r > 0 \; \exists \mathcal{A}$ bounded semi-analytic
$A \cap B(a, r) = \{x : (x, y) \in \mathcal{A}\}$

How about Hölder regularity ?

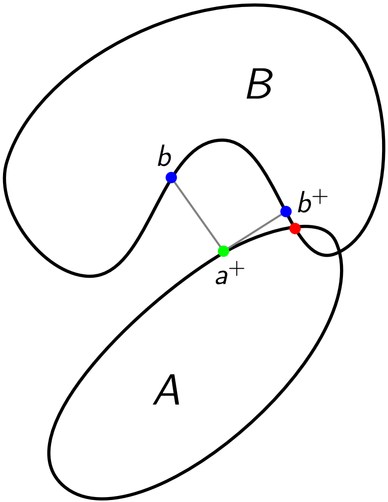


B convex

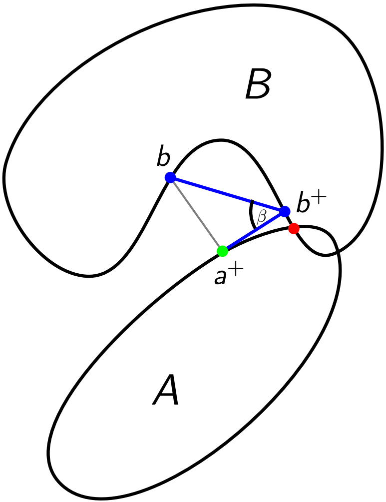


B convex

$\beta \geq 90°$

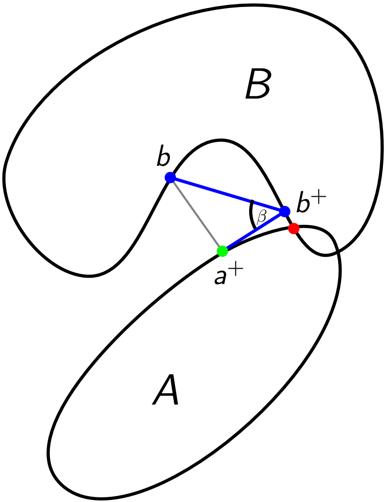


B non-convex



B non-convex

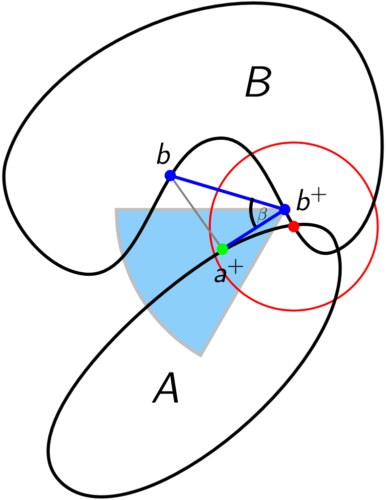
$\beta < 90^\circ$ possible



B non-convex

B super-regular :

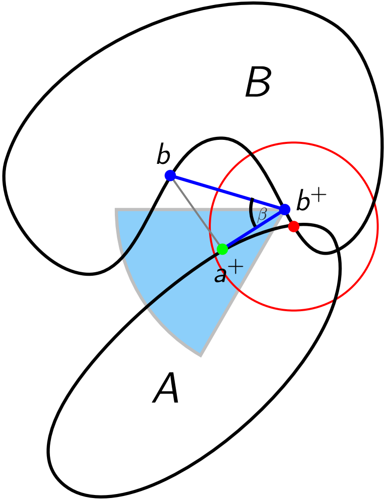
β not too small



B non-convex

B superregular :

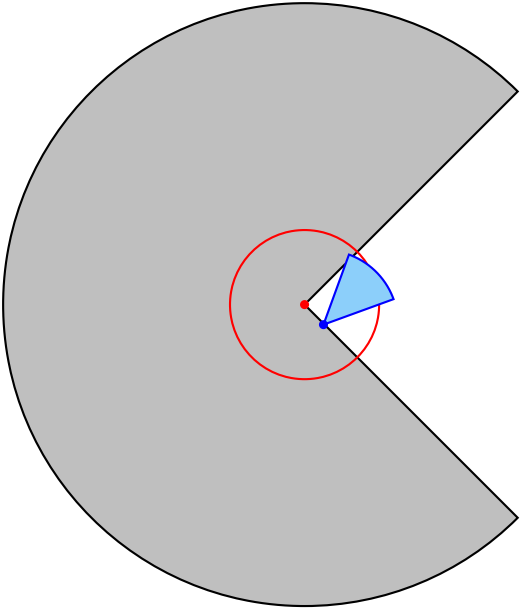
β not too small

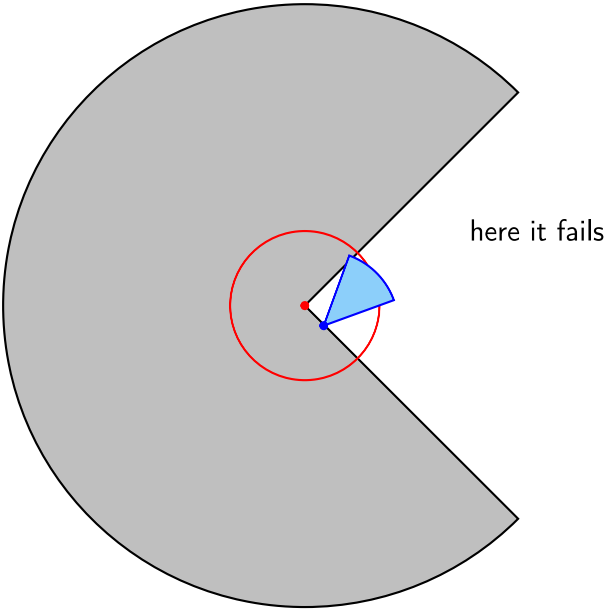


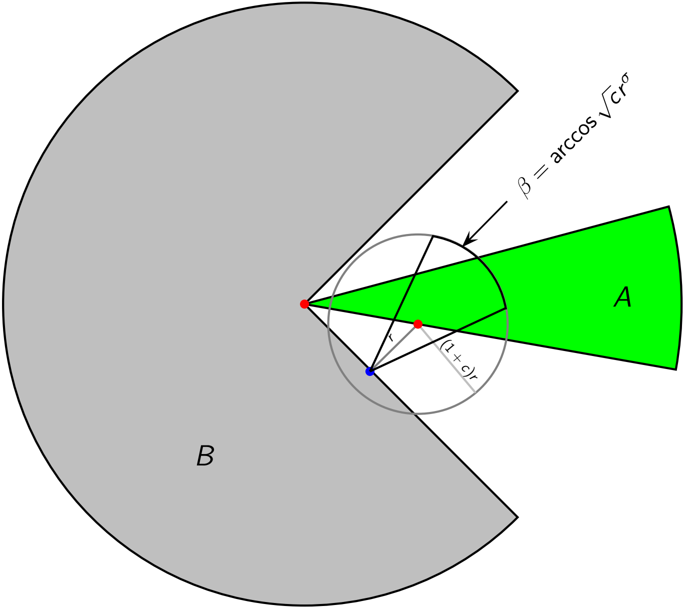
B non-convex

B superregular :

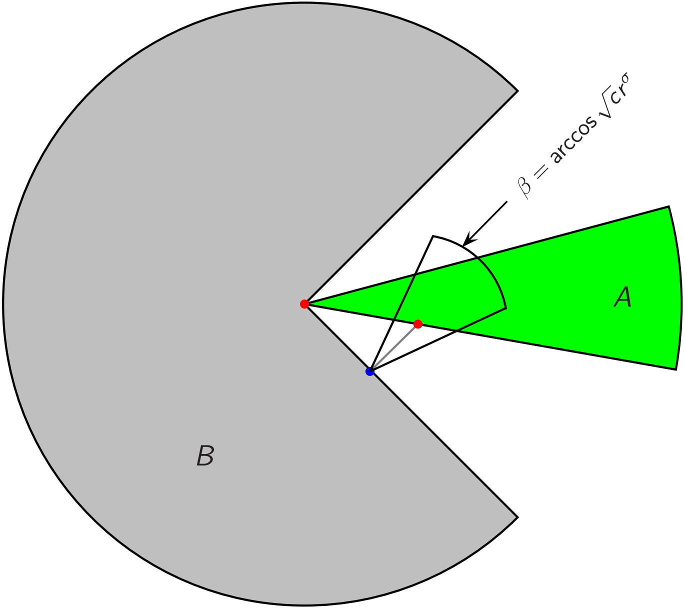
β not too small

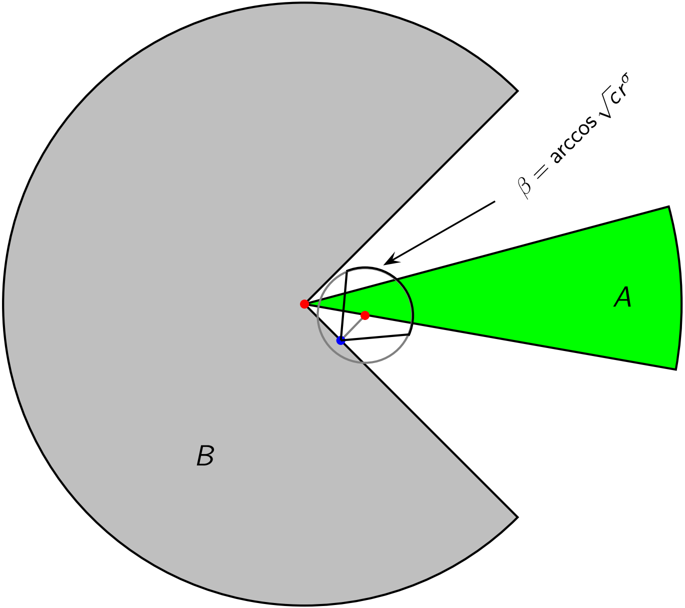



here it fails

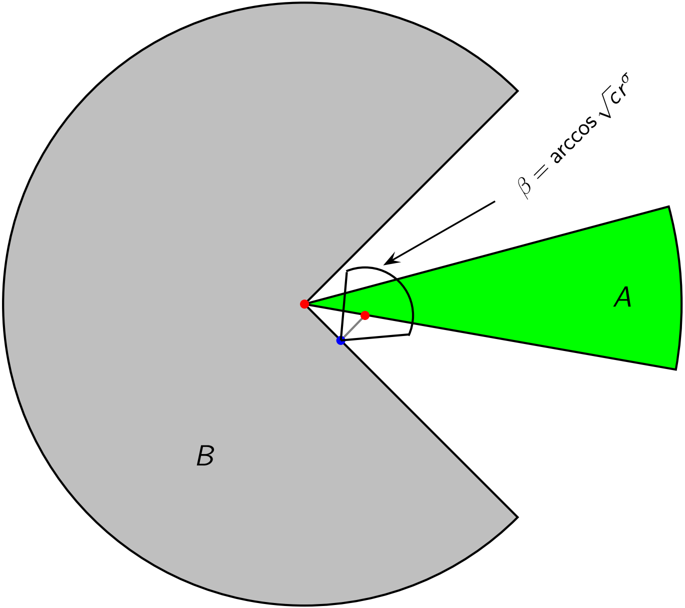


$\beta = \arccos\sqrt{cr^\sigma}$
A
B
r
$(1+c)r$


$\beta = \arccos\sqrt{cr^{\sigma}}$
A
B



$\beta = \arccos\sqrt{cr^{\sigma}}$
A
B



$\beta = \arccos\sqrt{cr^{\sigma}}$
A
B

Consequence : Our notion of Hölder regularity still in business for packman. Can enter into corners.

Corollary

(Noll, Rondepierre 2013). *Suppose* A, B *are sub-analytic, and* B *is Hölder regular with respect to* A. *Suppose the alternating sequence* a_k, b_k *is bounded and satisfies* $a_k - b_k \to 0$. *Then there exists* $\omega \in [0, 2)$ *such that it converges to a point* $a^* \in A \cap B$ *with speed*

$$\|a_k - a^*\| = \mathcal{O}\left(k^{-\frac{2-\omega}{2\omega}}\right), \qquad \|b_k - a^*\| = \mathcal{O}\left(k^{-\frac{2-\omega}{2\omega}}\right)$$

Application : Phase retrieval

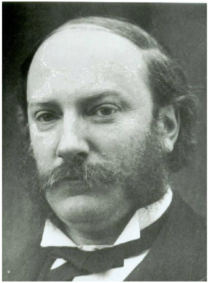
Phase retrieval

Reconstruct unknown signal $x(t)$, $t = 0, \ldots, N-1$ from known Fourier amplitude $a(f) = |\widehat{x}(f)|$, $f = 0, \ldots, N-1$.

- Retrieve unknown phase $\widehat{x}(f)/|\widehat{x}(f)|$, hence the name.
- Have to add prior information like known support of x in time domain : $x(t) = 0$ for $t \notin S$.
- Or additional measurements (Fourier amplitude from a second Fourier plane ; Gerchberg-Saxton 1972).

Phase retrieval in interferometry (optics) :

- First mentioned in a letter by Lord Rayleigh to A. Michelson in 1892.
- Impossibility to solve without prior information clearly stated.
- First numerical scheme : Gerchberg-Saxton algorithm 1972

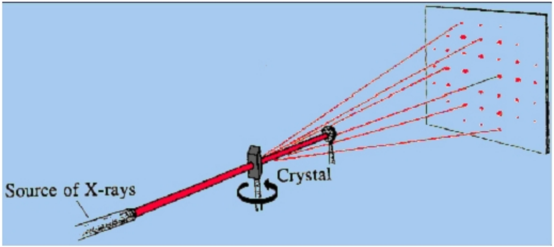


Lord Rayleigh
(early 1900s)

A. Michelson
(1907)

Some history

- Max von Laue (1912) proposes to use X-rays to visualise crystal structure via diffraction.
- David Sayre (1952) shows that non-periodic x can in principle also be retrieved from $|\widehat{x}|$ if $a = |\widehat{x}|$ is sampled twice the Nyquist rate in every dimension.
  - Deplores lack of methods to do it.
  - Hence participates in development of 1st `fortran` compiler.
- R.W. Gerchberg - O.W. Saxton (1972). 1st algorithm to retrieve x from $|\widehat{x}|$.
- J. Miao, P. Charalambous, J. Kirz, D. Sayre, Extending the methodology of X-ray crystallography to allow imaging of micrometre-sized non-crystalline specimens, *Nature* 400, 342-344, 1999. 45 years later Sayre is back !
- 2014. Individual proteins and nano-crystals can be visualized by CDI.



Max von Laue
(photo 1929)

David Sayre
(photo 1972)

W. O. Saxton
(photo 2012)

Gerchberg-Saxton error reduction (1972)

1. Given current estimate x compute $\widehat{x}$ and «correct» Fourier amplitude $\widehat{y}(f) := a(f)\dfrac{\widehat{x}(f)}{|\widehat{x}(f)|}$.
2. Take inverse Fourier transform y of $\widehat{y}$, and «correct» domain by putting $x^+(t) = \begin{cases} y(t) & \text{for } t \in S \\ 0 & \text{for } t \notin S \end{cases}$.
3. Replace x by x^+ and loop on.

$\Longrightarrow$ Optic, astronomy, crystallography, nano-materials, ...

$\Longrightarrow$ Cited 2643 times. Fourth most used algorithm ever

$\Longrightarrow$ No convergence proof since 1972.
We give the first.

Theorem

(Noll, Rondepierre 2013). *Suppose the phase retrieval problem has a solution $x^* \in A \cap B$. Suppose the physical domain constraint is represented by a sub-analytic set B. Then the Gerchberg-Saxton error reduction method converges in a neighborhood of x^* with speed of convergence $\mathcal{O}\left(k^{-\frac{2-\omega}{2\omega}}\right)$ for some $\omega \in (0,2)$.*

Proof. Equivalent to non-convex alternating projections :

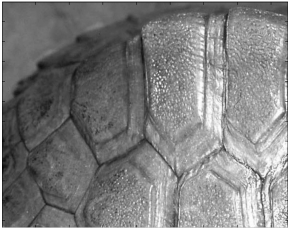
$$A = \{x \in \mathbb{C}^N : |\widehat{x}(f)| = a(f) \text{ for all } f\}$$

$$B = \{y \in \mathbb{C}^N : y(t) = 0 \text{ for all } t \notin S\}$$

$$P_A(x) = (a\widehat{x}/|\widehat{x}|)^{\widetilde{}} \qquad P_B(y) = y \cdot \mathbf{1}_S$$

□

Fourier phase and amplitude

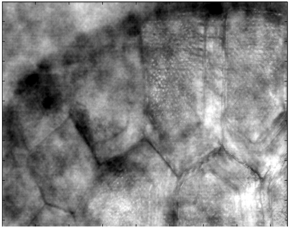


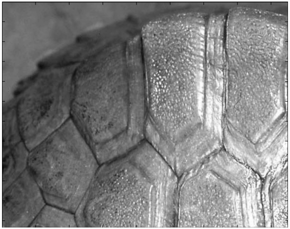
$r_t e^{i\phi_t}$

$r_w e^{i\phi_w}$

$r_t e^{i\phi_w}$

$r_w e^{\phi_t}$



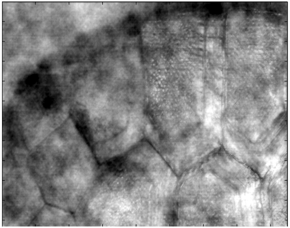


$r_t e^{i\phi_t}$

$r_w e^{i\phi_w}$

$r_t e^{i\phi_w}$

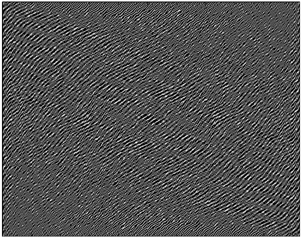
$r_w e^{\phi_t}$



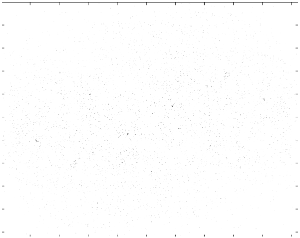
Consequences :

- Phase of Fourier transform $\widehat{x}/|\widehat{x}|$ gives the essential information about x.
- Amplitude of Fourier transform $|\widehat{x}|$ does not help to localize image x.
- Example : shift in time domain changes phase but not amplitude.
- Hence phase retrieval must be difficult. And it is !

original (unknown)

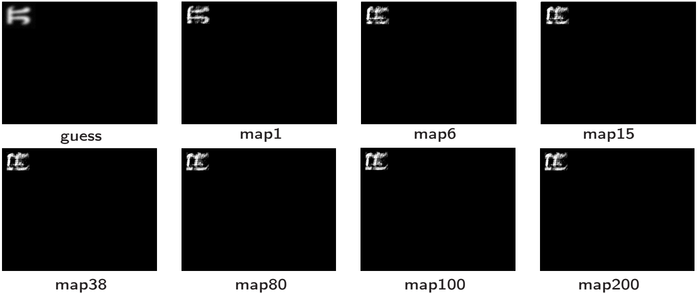


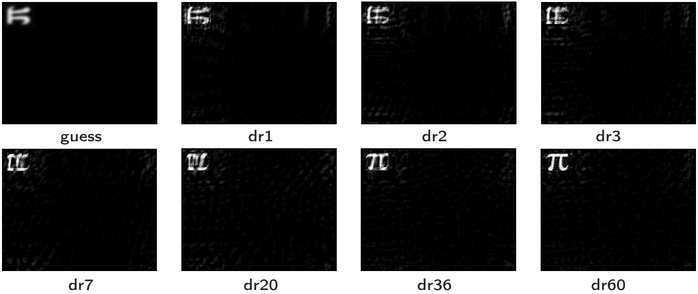
Fourier phase (unknown)



Fourier amplitude (known)

estimated support (prior)


guess
map1
map6
map15
map38
map80
map100
map200

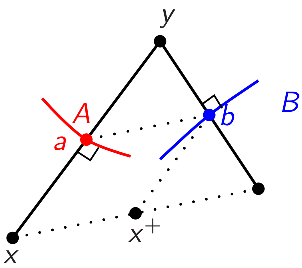

guess
dr1
dr2
dr3
dr7
dr20
dr36
dr60

- Ideal image x_0 is PI-image enlarged to size 1024×1024 by 0-padding.
- 0 is black, 256 is white.
- Initial guess is blurred and noisy version of the PI-image which is then rotated 90°.
- Fourier amplitude $a = |\widehat{x_0}|$ is known exactly.
- $A = \{x \in \mathbb{C}^{1024 \times 1024} : |\widehat{x}(f)| = a(f) \text{ for all frequencies } f\}$.
- $B = \{y \in \mathbb{C}^{1024 \times 1024} : y(t) = 0 \text{ for all pixels } t \text{ not in mask}\}$.
- Mask is gray region around the PI-symbol. Prior assumption is that values outside that mask equal 0.
- MAP does not fully succeed within reasonable time.
- Douglas-Rachford recovers phase quite nicely.

J. Douglas, H.H. Rachford. On the numerical solution of heat conduction problems in two and three dimensions. *TAMS* 82 (1956), 421 – 439.

P.-L. Lions, B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM J.Num. Anal.* 16 (1979), 946–979.

J.R. Fienup. Phase retrieval algorithms : a comparison. *Applied Optics*, 1982. HIO = Hybrid-Input-Output



$a \in P_A(x)$
$y = 2a - x \in R_A(x)$
$b \in P_B(y)$
$x^+ = x + b - a$

reflect-reflect-average

Convergence for non-convex alternating projections

A.S. Lewis, J. Malick (2008). Alternating projections on manifolds. *Math. Oper. Res.*

A.S. Lewis, R. Luke, J. Malick (2009). Local linear convergence for alternating and averaged non-convex projections. *Foundations Comp. Math.*

H.H. Bauschke, D.R. Luke, H.M. Phan, X. Wang (2013). Restricted normal cones and the method of alternating projections. *Set-valued Var. Anal.*

D. Noll, A. Rondepierre (2013). On local convergence of the method of alternating projections. Foundations of Computational Mathematics, 2015.

Convergence for non-convex Douglas-Rachford

R. Hesse, D.R. Luke (2013). Nonconvex notions of regularity and convergence of fundamental algorithms for feasibility problems, *SIAM Journal on Optimization* 23(4), 2397–2419.

H.H. Bauschke, D. Noll (2014). On the local convergence of the Douglas–Rachford algorithm. *Archiv der Mathematik* 102, 589 – 600.

H.M. Phan (2014). Linear convergence of the Douglas-Rachford method for two closed sets. *Optimzation* 2015.

Pointer to Actual News

Comparison of resolution of CDI with :

Chemistry Nobel Price 2014 : Fluorescence Microscopy

Stefan Hell
*1962

William Moerner
*1953

Eric Betzig
*1960

Fluorescence Microscopy : $1\mu\text{m} = 10^{-6}\text{m}$

CDI : $10\text{nm} = 10^{-8}\text{m}$ (organic)
$2\text{ nm} = 2 \cdot 10^{-9}\text{m}$ (anorganic)

Thanks for your attention !

Application : EM-algorithm

A.P. Dempster, N.M. Laird, D.B. Rubin. Maximum likelihood from incomplete data via the EM-algorithm.
J. Royal Stat. Soc. Series B, vol. 39, no. 1 (1977), 1 – 38.

$\Longrightarrow$ Cited 38230 times since 1977
$\Longrightarrow$ However, convergence proof incorrect.
$\Longrightarrow$ Since then only proofs for specific situations.

Our approach gives the first local convergence proof for Gaussian laws when parameter set is not convex

Structured low-rank matrix approximation

Given a structured matrix $x \in \mathcal{S}$, solve the problem

$$(P) \quad \begin{array}{ll} \text{minimize} & \|x' - x\|_F \\ \text{subject to} & x' \in \mathcal{S} \\ & \mathrm{rank}(x') \leq r \end{array}$$

- $\mathcal{S} =$ Hankel matrices (denoising of time series)
- $\mathcal{S} =$ Toeplitz matrices (spectral estimation problems)
- $\mathcal{S} =$ positive semidefinite matrices
- $\mathcal{S} =$ stable matrices

Use non-convex alternating projections :

$$A = \{x : x \in \mathcal{S}\} \qquad P_A = \text{easy ???}$$

$$B = \{x : \mathrm{rank}(x) \leq r\} \qquad P_B = \text{truncated SVD}$$

Can now prove local convergence to $x^* \in \{x : \mathrm{rank}(x) \leq r, x \in \mathcal{S}\}$. Need not be solution of (P)

Sparse affine feasibility

$$\begin{array}{ll} \text{minimize} & \|x\|_0 = \text{number of non-zero entries in } x \\ \text{subject to} & Ax = b \\ & x \in \mathbb{R}^n \end{array}$$

Use non-convex alternating projections :

$$A = \{x \in \mathbb{R}^n : \|x\|_0 \leq k\} = \bigcup_{\mathrm{card}(I) \leq k} \underbrace{\mathrm{span}\{e_i : i \in I\}}_{=:A_I}$$

$$P_A(x) = \bigcup_{I_{\text{active}}} P_{A_I}(x)$$

$$B = \{x \in \mathbb{R}^n : Ax = b\} \qquad P_B(x) = x - A^{\dagger}(Ax - b)$$

Packings in Grassmannian manifolds

$\mathbb{G}(k, \mathbb{C}^d)$ = all k-dimensional subspaces of $\mathbb{C}^d$

Represent $\underline{S} \in \mathbb{G}(k, \mathbb{C}^d)$ by unitary $S \in \mathbb{C}^{k \times d}$: $S^* S = I_k$, range$(S) = \underline{S}$

For two subspaces S, T do SVD :

$$S^* T = UCV^*$$

then $c_{kk} = \cos\theta_k$ the principal angles between $\underline{S}, \underline{T}$. Leads to distances between $\underline{S}$ and $\underline{T}$:

- Chordal distance : $\sqrt{\sin^2\theta_1 + \cdots + \sin^2\theta_k} = \left(k - \|S^* T\|_F^2\right)^{1/2}$
- Spectral distance : $\min_i \sin\theta_i = \left(k - \|S^* T\|_{2,2}^2\right)^{1/2}$
- Fubini-Study distance : $\arccos\left(\Pi_j \cos\theta_j\right)$
- Geodesic distance : $\sqrt{\theta_1^2 + \cdots + \theta_k^2}$

Packing for the chordal distance :

$$\text{pack}(S_1, \ldots, S_N) := \min_{m \neq n} d_{\mathrm{chord}}(S_m, S_n) = \min_{m \neq n} \left(k - \|S_m^* S_n\|_F^2\right)^{1/2}$$

True problem : $\max\limits_{\{S_1, \ldots, S_N\}} \mathrm{pack}_{\mathrm{chord}}(S_1, \ldots, S_N)$

Instead feasibility problem : Given $\rho > 0$, want $\{S_1, \ldots, S_N\}$ such that $\mathrm{pack}_{\mathrm{chord}}(S_1, \ldots, S_N) \geq \rho$

$$\min_{m \neq n} \left(k - \|S_m^* S_n\|_F^2\right)^{1/2} \geq \rho \equiv \max_{m \neq n} \|S_m^* S_n\|_F \leq \mu := \sqrt{k - \rho^2}$$

Put $S := [S_1 S_2 \dots S_N]$

Gramian $G = S^* S \in \mathbb{C}^{kN \times kN} \succeq 0$, $G_{mn} = S_m^* S_n$

1. G is Hermitian
2. Each diagonal block of G is identity I_k
3. $G \succeq 0$
4. $\mathrm{rank}(G) \leq d$
5. $\mathrm{trace}(G) = kN$

Conversely, any G with these properties can be factored $G = S^* S$ and $S = [S_1 \dots S_N]$ gives then rise to a configuration of N subspaces in the Grassmannian $\mathbb{G}(k, \mathbb{C}^d)$.

Now alternating projections :

Structural constraint (convex) :

$$A = \{H \in \mathbb{C}^{kN \times kN} : H = H^*, H_{nn} = I_k, \|H_{mn}\|_F \leq \mu\}$$

Spectral constraint (non-convex) :

$$B = \{G \in \mathbb{C}^{kN \times kN} : G \succeq 0, \mathrm{rank}(G) \leq d, \mathrm{trace}(G) = kN\}$$

Any solution $G \in A \cap B$ gives a packing of size N of the Grassmannian with chordal distance packing index $\geq \rho$.

Compute projection on A (easy) :

$$H = P_A(G) : H_{mn} = \begin{cases} G_{mn} & \|G_{mn}\|_F \leq \mu \\ \mu G_{mn}/\|G_{mn}\|_F & \text{else} \end{cases}$$

Compute projection on B (more involved but possible) :

Let $H = \sum_{j=1}^{kN} \lambda_j u_j u_j^*$ be spectral decomposition, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{kN}$. Then

$$G = \sum_{j=1}^{d} (\lambda_j - \gamma)_+ u_j u_j^* \in P_B(H)$$

provided γ is chosen such that $\sum_{j=1}^{d} (\lambda_j - \gamma)_+ = kN$.