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# Non-polyhedral extensions of the Frank-and-Wolfe theorem

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**Abstract**
In 1956 Marguerite Frank and Paul Wolfe proved that a quadratic function which is bounded below on a polyhedron $P$ attains its infimum on $P$. In this work we search for larger classes of sets $F$ with this Frank-and-Wolfe property. We establish the existence of non-polyhedral Frank-and-Wolfe sets, obtain internal characterizations by way of asymptotic properties, and investigate stability of the Frank-and-Wolfe class under various operations.

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Quadratic optimization — asymptotes — Motzkin-sets — Frank-and-Wolfe theorem

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## 1. Introduction

In this paper we investigate extensions of the famous Frank and Wolfe theorem [8, 5, 6, 7, 1], which states that a quadratic function $f$ which is bounded below on a closed convex polyhedron $P$ attains its infimum on $P$. This has applications to linear complementarity problems, and a natural question is whether this property is shared by larger classes of non-polyhedral convex sets $F$.

The present work expands on [14], where the Frank-and-Wolfe property was successfully related to asymptotic properties of a set $F$. Following this line, we presently obtain a complete characterization of the Frank-and-Wolfe property within the class of Motzkin decomposable sets. In particular, the converse of a result of Kummer [12] is obtained.

A second theme addresses versions of the Frank-and-Wolfe theorem where the class of quadratic functions is further restricted. One may for instance ask for sets $F$ on which convex or quasi-convex quadratics attain their finite infima. It turns out that this class has a complete characterization as those sets which have no flat asymptotes in the sense of Klee. As a consequence we obtain a version of the Frank-and-Wolfe theorem which extends a result of Rockafellar [16, Sect. 27] and Belousov and Klatte [4] on convex polynomials.

Invariance of Frank-and-Wolfe type sets under various operations such as finite intersections, unions, cross-products, sums, and under affine images and pre-images are also investigated.

The structure of the chapter is as follows. In section 2 we give the definition and collect basic information on $FW$-sets. In section 3 we consider quasi-Frank-and-Wolfe sets, where a version of the Frank and Wolfe theorem for quasi-convex quadratics is discussed. It turns out that the same class allows many more applications, as it basically suffices to have polynomial functions which have at least one convex sub-level set. In section 4 we consider sets with a generalized Motzkin decomposition of the form $F = K + D$ with $K$ compact and $D$ a closed convex cone. This class was used by Kummer [12], who proved a version of the Frank and Wolfe theorem in this class when $D$ is polyhedral. We give a new proof of this result and also establish its inverse, that is, if a Motzkin set satisfies the Frank and Wolfe theorem, then the cone $D$ must be polyhedral. Section 5 discusses invariance properties of the class of Motzkin sets with the Frank and Wolfe property.

## Notations

We generally follow Rockafellar's book [16]. The closure of a set $F$ is $\overline{F}$. The Euclidean norm in $\mathbb{R}^n$ is $\|\cdot\|$, and the Euclidean distance is $\mathrm{dist}(x,y) = \|x-y\|$. For subsets $M,N$ of $\mathbb{R}^n$ we write $\mathrm{dist}(M,N) = \inf\{\|x-y\| : x \in M, y \in N\}$. A direction $d$ with $x+td \in F$ for every $x \in F$ and every $t \geq 0$ is called a direction of recession of $F$, and the cone of all directions of recession is denoted as $0^+F$.

A function $f(x) = \frac{1}{2}x^\mathsf{T}Ax + b^\mathsf{T}x + c$ with $A = A^\mathsf{T} \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ is called quadratic. The quadratic $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-convex on a convex set $F \subset \mathbb{R}^n$ if the sub-level sets of $f_{|F} : F \to \mathbb{R}$ are convex. Similarly, $f$ is convex on the set $F$ if $f_{|F}$ is convex.

## 2. Frank-and-Wolfe sets

The following definition is the basis for our investigation:

**Definition 1.** A set $F \subset \mathbb{R}^n$ is called a *Frank-and-Wolfe set*, for short a $FW$-set, if every quadratic function $f$ which is bounded below on $F$ attains its infimum on $F$.

In [14] this notion was introduced for convex sets $F$, but in the present note we extend it to arbitrary sets, as this property is not really related to convexity. The classical Frank-and-Wolfe theorem says that every closed convex polyhedron is a $FW$-set, cf. [8, 5, 6, 7]. Here we are interested in identifying and characterizing more general classes of sets with this property. We start by collecting some basic information about $FW$-sets.

**Proposition 1.** *Affine images of FW-sets are again FW-sets.*

*Proof.* Let $F$ be a $FW$-set in $\mathbb{R}^n$ and and $T : \mathbb{R}^n \to \mathbb{R}^m$ an affine mapping. We have to show that $T(F)$ is a $FW$-set. Let $f$ be a quadratic on $\mathbb{R}^m$ which is bounded below on $T(F)$, then $f \circ T$ is a quadratic on $\mathbb{R}^n$, which is bounded below on $F$, hence attains its infimum at some $x \in F$. Then $f$ attains its infimum at $Tx \in T(F)$. □

It is equally easy to see that every $FW$-set is closed, because if $x \in \overline{F}$, then the quadratic function $f = \|\cdot - x\|^2$ has infimum 0 on $F$, and if this infimum is to be attained, then $x \in F$. As a consequence, a bounded set $F$ is $FW$ iff it is closed, so there is nothing interesting to report on bounded $FW$-sets, and the property is clearly aimed at the analysis of unbounded sets.

One can go a little further than just proving closedness of $FW$-sets and get first information about their asymptotic behavior. We need the following:

**Definition 2.** An affine manifold $M$ in $\mathbb{R}^n$ is called an *$f$-asymptote* of the set $F \subset \mathbb{R}^n$ if $F \cap M = \emptyset$ and $\operatorname{dist}(F,M) = 0$.

This expands on Klee [11], who introduced this notion for convex sets $F$. The symbol $f$ stands for *flat* asymptote. This allows us now to propose the following:

**Proposition 2.** *Let $F$ be a FW-set. Then $F$ has no f-asymptotes.*

*Proof.* Let $M$ be an affine subspace such that $\operatorname{dist}(F,M) = 0$. We have to show that $M \cap F \neq \emptyset$. Let $M = \{x \in \mathbb{R}^n : Ax - b = 0\}$ for a suitable matrix $A$ and vector $b$. Put $f(x) = \|Ax - b\|^2$, then $f$ is quadratic, and $\gamma = \inf\{f(x) : x \in F\} \geq 0$. Now there exist $x_n \in F$ and $y_n \in M$ with $\operatorname{dist}(x_n, y_n) \to 0$. But $Ay_n = b$, and $\|A(x_n - y_n)\| \leq \|A\|\|x_n - y_n\| \to 0$, hence $Ax_n \to b$, which implies $\gamma = 0$. Now since $F$ is a $FW$-set, this infimum is attained, hence there exists $x \in F$ with $f(x) = 0$, which means $Ax = b$, hence $x \in M$. That shows $F \cap M \neq \emptyset$, so $M$ is not an $f$-asymptote of $F$. □

**Remark 1.** An immediate consequence of Propositions 1, 2 is that affine images of $FW$-sets, and in particular, projections of $FW$-sets, are always closed.

Yet another trivial fact is the following:

**Proposition 3.** *Finite unions of FW-sets are FW.* □

We conclude this preparatory section by looking at invariance of the $FW$-class under affine pre-images. First we need the following:

**Proposition 4.** *If $F \subset \mathbb{R}^n$ is a FW-set and $M \subset \mathbb{R}^m$ is an affine manifold, then $F \times M$ is a FW-set in $\mathbb{R}^n \times \mathbb{R}^m$.*

*Proof.* Since translates of $FW$-sets are $FW$-sets, we may assume that $M$ is a linear subspace, and then there is no loss of generality in assuming that $M = \mathbb{R}^m$. Moreover, by an easy induction argument, we only need to consider the case when $m = 1$.

Let $q$ be a quadratic function on $\mathbb{R}^n \times \mathbb{R}$ bounded below on $F \times \mathbb{R}$. We can write $q(x,t) = \frac{1}{2}x^T Ax + \frac{1}{2}bt^2 + tc^T x + d^T x + et + f$ for suitable $A$, $b$, $c$, $d$, $e$ and $f$. Clearly, $b \geq 0$, as otherwise $q$ could not be bounded below on $F \times \mathbb{R}$. Now we have $\inf_{(x,t)\in F\times\mathbb{R}} q(x,t) = \inf_{x\in F} \inf_{t\in\mathbb{R}} q(x,t)$.

First consider the case $b > 0$. Then the inner infimum in the preceding expression is attained at $t = -\frac{c^T x + e}{b}$. Hence we have $\inf_{(x,t)\in F\times\mathbb{R}} q(x,t) = \inf_{x\in F} q\left(x, -\frac{c^T x+e}{b}\right)$. Given that $q\left(x, -\frac{c^T x+e}{b}\right)$ is a quadratic function of $x$ and is obviously bounded below on $F$, it attains its infimum over $F$ at some $\bar{x} \in F$. Therefore $q$ attains its infimum over $F \times \mathbb{R}$ at $\left(\bar{x}, -\frac{c^T \bar{x}+e}{b}\right)$.

Now consider the case $b = 0$, $c \neq 0$. Here $F$ must be contained in the hyperplane $c^T x + e = 0$. Substituting this, we get $\inf_{(x,t)\in F\times\mathbb{R}} q(x,t) = \inf_{x\in F} \left\{\frac{1}{2}x^T Ax + d^T x\right\} + f$. Hence, the quadratic function given by $\frac{1}{2}x^T Ax + d^T x$ is bounded below on $F$ and, for every minimizer $\bar{x} \in F$ and every $t \in \mathbb{R}$, the point $(\bar{x}, t)$ is a minimizer of $q$ over $F \times \mathbb{R}$.

Finally, when $b = 0$, $c = 0$ it follows that we must also have $e = 0$, so $q$ no longer depends on $t$, and we argue as in the previous case. □

**Remark 2.** As we shall see in the next section (example 1), the cross product $F_1 \times F_2$ of two $FW$-sets $F_i$ is in general no longer a $FW$-set, so Proposition 4 exploits the very particular situation.

We have the following consequence:

**Corollary 1.** *Let $F$ be a FW-set in $\mathbb{R}^n$ and $M$ an affine subspace of $\mathbb{R}^n$. Then $F+M$ is a FW-set.*

*Proof.* $F \times M$ is a $FW$-set by Proposition 4, and its image under the mapping $(x,y) \to x+y$ is a $FW$-set by Proposition 1, and that set is $F+M$. □

Concerning pre-images, we have the following consequence of Proposition 4:

**Proposition 5.** *Let $T$ be an affine operator and suppose the $FW$-set $F$ is contained in the range of $T$. Then $T^{-1}(F)$ is a $FW$-set.*

*Proof.* Since the notion of a $FW$-set is invariant under translations and under coordinate changes, we can assume that $T$ is a surjective linear operator $T : \mathbb{R}^n \to \mathbb{R}^m$ and $F \subset \mathbb{R}^m$. Now $\widetilde{F} = (T_{|\ker(T)^\perp})^{-1}(F)$ is an affine image of the $FW$-set $F$, hence by Proposition 1 is a $FW$-subset of $\mathbb{R}^n$. By Corollary 1 the set $\widetilde{F} + \ker(T)$ is a $FW$-set, but this set is just $T^{-1}(F)$. □

**Remark 3.** It is not clear whether this result remains true when $F$ is not entirely contained in the range of $T$, i.e., when only $F \cap \mathrm{range}(T) \neq \emptyset$. In contrast, see Corollary 3 and Proposition 9.

More sophisticated invariance properties of the class of $FW$-sets will be investigated later. For instance, one may ask whether or under which conditions finite intersections, cartesian products, or closed subsets of $FW$-sets are again $FW$.

## 3. Frank-and-Wolfe theorems for restricted classes of quadratic functions

Following [14] it is also of interest to investigate versions of the Frank and Wolfe theorem, where the class of quadratic functions is further restricted. The following notion is from [14]:

**Definition 3.** A convex set $F \subset \mathbb{R}^n$ is called a *quasi-Frank-and-Wolfe set*, for short a *qFW*-set, if every quadratic function $f$, which is quasi-convex on $F$ and bounded below on $F$, attains its infimum on $F$.

Note that for the class of *qFW*-sets we have to maintain convexity as part of the definition, as otherwise absurd situations might occur, so the notion is precisely as introduced in [14].

**Remark 4.** Every convex *FW*-set is clearly a *qFW*-set. The converse is not true, i.e., *qFW*-sets need not be *FW*, as will be seen in Example 1. It is again clear that *qFW*-sets are closed, and that affine images of *qFW*-sets are *qFW*.

It turns out that $f$-asymptotes are the key to understanding the quasi-Frank-and-Wolfe property. We have the following:

**Theorem 1.** *Let $F$ be a convex set in $\mathbb{R}^n$. Then the following statements are equivalent:*

(1) *Every polynomial $f$ which has at least one nonempty convex sub-level set on $F$ and which is bounded below on $F$ attains its infimum on $F$.*

(2) *$F$ is a qFW-set.*

(3) *Every quadratic function $q$ which is convex on $F$ and bounded below on $F$ attains its infimum on $F$.*

(4) *$F$ has no f-asymptotes.*

(5) *$T(F)$ is closed for every affine mapping $T$.*

(6) *$P(F)$ is closed for every orthogonal projection $P$.*

*Proof.* The implication (1) $\Longrightarrow$ (2) is clear, because for a quasi-convex function on $F$ *every* sub-level set on $F$ is convex. The implication (2) $\Longrightarrow$ (3) is also evident. Implication (3) $\Longrightarrow$ (4) follows immediately with the same proof as Proposition 2, because the quadratic $f(x) = \|Ax-b\|^2$ used there is convex.

Let us prove (4) $\Longrightarrow$ (5). We may without loss of generality assume that $T$ is linear, as properties (4) and (5) are invariant under translations. Suppose $T(F)$ is not closed and pick $y \in \overline{T(F)} \setminus T(F)$. Put $M = T^{-1}(y)$, then $M$ is an affine manifold. Note that $M \cap F = \emptyset$, because $T(M) = \{y\}$. Now pick $y_n \in T(F)$ such that $y_n \to y$, and choose $x_n \in T^{-1}(y_n) \cap F$. Since $T$ is affine there exist $x_n' \in T^{-1}(y_n)$ such that $x_n' \to x' \in T^{-1}(y)$. We have $\|x_n - (x' - x_n' + x_n)\| \to 0$, with $x_n \in F$, and since $x_n - x_n' \in \ker(T)$, we have $x' - x_n' + x_n \in x' + \ker(T) = M$. That proves $\operatorname{dist}(F,M) = 0$, and so $F$ has $M$ as an $f$-asymptote, a contradiction.

The implication (5) $\Longrightarrow$ (6) is clear. Let us prove (6) $\Longrightarrow$ (1). We will prove this by induction on $n$. For $n = 1$ the implication is clearly true, because any polynomial $f : \mathbb{R} \to \mathbb{R}$ which is bounded below on a convex set $F \subset \mathbb{R}$ satisfying (6) attains its infimum on $F$, as (6) implies that $F$ is closed. Suppose therefore that the result is true for dimension $n-1$, and consider a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ which is bounded below on a set $F \subset \mathbb{R}^n$ with property (6) such that $S_\alpha := \{x \in F : f(x) \le \alpha\}$ is nonempty and convex for some $\alpha \in \mathbb{R}$. We may without loss of generality assume that the dimension of $F$ is $n$, i.e., that $F$ has nonempty interior, as otherwise $F$ is contained in a hyperplane, and then the result follows directly from the induction hypothesis. If $\alpha = \gamma := \inf\{f(x) : x \in F\}$, then $f$ clearly attains $\alpha$, so we assume from now on that $\alpha > \gamma$. If $S_\alpha := \{x \in F : f(x) \le \alpha\}$ is bounded, then by the Weierstrass extreme value theorem the infimum of $f$ over $S_\alpha$ is attained, because by hypothesis (6) the set $F$ is closed. But this infimum is also the infimum of $f$ over $F$, so in this case we are done. Assume therefore that $S_\alpha$ is unbounded. Since $S_\alpha$ is a closed convex set, it has a direction of recession $d$, that is, $x + td \in S_\alpha$ for every $t \ge 0$ and every $x \in S_\alpha$. Fix $x \in S_\alpha$. This means

$$\gamma \le f(x+td) \le \alpha$$

for every $t \ge 0$. Since $t \mapsto f(x+td)$ is a polynomial on the real line, which is now bounded on $[0,\infty)$, it must be constant as a function of $t$, so that $f(x) = f(x+td)$ for all $t \ge 0$, and then clearly also $f(x+td) = f(x)$ for every $t \in \mathbb{R}$. But the argument is valid for every $x \in S_\alpha$. By assumption $F$ has dimension $n$, so $S_\alpha$ has nonempty interior. That shows $f(x+td) = f(x)$ for all $x$ in a nonempty open set contained in $S_\alpha$ and all $t \in \mathbb{R}$. Altogether, since $f$ is a polynomial, we obtain

$$f(x+td) = f(x) \text{ for every } x \in \mathbb{R}^n \text{ and every } t \in \mathbb{R}. \tag{1}$$

Now let $P$ be the orthogonal projection onto the hyperplane $H = d^{\perp}$. Then $\widetilde{f} := f_{|H}$ is a polynomial on the $(n-1)$-dimensional space $H$ and takes the same values as $f$ due to (1). In particular, $\widetilde{f} = f_{|H}$ is bounded below on the set $\widetilde{F} = P(F)$.

We argue that the induction hypothesis applies to $\widetilde{F}$. Indeed, $\widetilde{F}$ being the image of $F$ under a projection, is closed by condition (6). Its dimension is $n-1$, and moreover, every projection of $\widetilde{F}$ is closed, because any such projection is also a projection of $F$.

It remains to prove that the restriction of $\widetilde{f}$ to $\widetilde{F}$ has a nonempty convex sub-level set. To this end it will suffice to prove that, for $\widetilde{S}_{\alpha} := \{x \in \widetilde{F} : \widetilde{f}(x) \leq \alpha\}$, one has $\widetilde{S}_{\alpha} = P(S_{\alpha})$. This will easily follow from the observation that $\widetilde{f} \circ P = f$, which is an immediate consequence of (1). Let $x \in \widetilde{S}_{\alpha}$. Since $x \in \widetilde{F}$, we have $P(x') = x$ for some $x' \in F$, and hence $f(x') = \left(\widetilde{f} \circ P\right)(x') = \widetilde{f}(P(x')) = \widetilde{f}(x) \leq \alpha$, which proves that $x' \in S_{\alpha}$. Therefore $x \in P(S_{\alpha})$, which shows $\widetilde{S}_{\alpha} \subset P(S_{\alpha})$. To prove the opposite inclusion, let $x \in P(S_{\alpha})$. We then have $x = P(x')$ for some $x' \in S_{\alpha}$. From the inclusion $S_{\alpha} \subset F$, it follows that $x \in P(F) = \widetilde{F}$. On the other hand, $\widetilde{f}(x) = f(x') \leq \alpha$. This shows $x \in \widetilde{S}_{\alpha}$ and proves the inclusion $P(S_{\alpha}) \subset \widetilde{S}_{\alpha}$ and hence our claim $\widetilde{S}_{\alpha} = P(S_{\alpha})$.

Altogether, $\widetilde{f}$ now attains its infimum on $\widetilde{F}$ by the induction hypothesis, and then $f$, having the same values, also attains its infimum on $F$. This proves the validity of (1). □

**Remark 5.** The equivalence of (4) and (6) can already be found in [11].

**Remark 6.** All that matters in condition (1) is the *rigidity* of polynomials. Any class $\mathscr{F}(L)$ of continuous functions defined on affine subspaces $L$ of $\mathbb{R}^n$ with the following properties would work as well: (i) $\mathscr{F}(L)$ is defined for every $L \subset \mathbb{R}^n$ and every $n$. (ii) If $f \in \mathscr{F}(\mathbb{R})$ is bounded below on a closed interval on $\mathbb{R}$, then $f$ attains its infimum. (iii) If $f \in \mathscr{F}(\mathbb{R}^n)$ and $H$ is a hyperplane in $\mathbb{R}^n$, then $f_{|H} \in \mathscr{F}(H)$. (iv) If $f \in \mathscr{F}(\mathbb{R}^n)$ is bounded (above and below) on some ray $x + \mathbb{R}^+ d \subset \mathbb{R}^n$, then $f$ does not depend on $d$, i.e., $f(x) = f(x+td)$ for all $t \in \mathbb{R}$.

We had seen in section 2 that *FW*-sets have no $f$-asymptotes. Moreover, from the results of this section we see that if $F$ is convex and has no $f$-asymptotes, then it is already a *qFW*-set. This rises the question whether the absence of $f$-asymptotes also serves to characterize *FW*-sets, or if not, whether it does so at least for convex $F$. We indicate by way of two examples that this is not the case, i.e., the absence of $f$-asymptotes does *not* characterize Frank-and-Wolfe sets. Or put differently, there exist quasi-Frank-and-Wolfe sets which are not Frank-and-Wolfe.

**Example 1.** We construct a closed convex set $F$ without $f$-asymptotes, which is not Frank-and-Wolfe. We use Example 2 of [13], which we reproduce here for convenience. Consider the optimization program

$$\begin{array}{ll} \text{minimize} & q(x) = x_1^2 - 2x_1x_2 + x_3x_4 + 1 \\ \text{subject to} & c_1(x) = x_1^2 - x_3 \leq 0 \\ & c_2(x) = x_2^2 - x_4 \leq 0 \\ & x \in \mathbb{R}^4 \end{array}$$

then as Lou and Zhang [13] show the constraint set $F = \{x \in \mathbb{R}^4 : c_1(x) \leq 0, c_2(x) \leq 0\}$ is closed convex, and the quadratic function $q$ has infimum $\gamma = 0$ on $F$, but this infimum is not attained.

Let us show that $F$ has no $f$-asymptotes. Note that $F = F_1 \times F_2$, where $F_1 = \{(x_1, x_3) \in \mathbb{R}^2 : x_1^2 - x_3 \leq 0\}$, $F_2 = \{(x_2, x_4) \in \mathbb{R}^2 : x_2^2 - x_4 \leq 0\}$. Observe that $F_1 \cong F_2$, and that $F_1$ does not have asymptotes, being a parabola. Therefore, $F$ does not have $f$-asymptotes either. This can be seen from the following:

**Proposition 6.** *Any nonempty finite intersection of qFW-sets is again a qFW-set.*

*Proof.* By Theorem 1 the result follows immediately from a theorem of Klee [11, Thm. 4], which says that finite intersections of sets without $f$-asymptotes have no $f$-asymptotes. ☐

**Corollary 2.** *If $F_1,\dots,F_m$ are qFW-sets, then the cartesian product $F_1\times\cdots\times F_m$ is again a qFW-set.*

*Proof.* Consider for the ease of notation the case of two sets $F_i\subset\mathbb{R}^{d_i}$, $i=1,2$. Then write

$$F_1\times F_2=\left(F_1\times\mathbb{R}^{d_2}\right)\cap\left(\mathbb{R}^{d_1}\times F_2\right).$$

Now $F_1\times\mathbb{R}^{d_2}$ is also *qFW*, and so is $\mathbb{R}^{d_1}\times F_2$, and hence the result follows from Proposition 6. The fact that $F_1\times\mathbb{R}^{d_1}$ is *qFW* is easily seen as follows: If $M$ is a $f$-asymptote of $F_1\times\mathbb{R}^{d_1}$, then $L=\{x:(x,y)\in M \text{ for some } y\}$ is a $f$-asymptote of $F_1$. ☐

**Remark 7.** Example 1 also tells us that the sum of $FW$-sets need not be a $FW$-set even when closed, as follows from the identity $F_1\times F_2=(F_1\times\{0\})+(\{0\}\times F_2)$. Note that even though $F_1\times F_2$ fails to be $FW$, it remains $qFW$ due to Corollary 2.

**Example 2.** Let $F$ be the epigraph of $f(x)=x^2+\exp(-x^2)$ in $\mathbb{R}^2$. Then $q(x,y)=y-x^2$ is bounded below on $F$, but does not attain its infimum, so $F$ is not *FW*. However, $F$ has no $f$-asymptotes, so it is *qFW*. ☐

**Remark 8.** In [14] it is shown explicitly that the ice-cream cone is not *qFW*. Here is a simple synthetic argument. The ice cream cone $D\subset\mathbb{R}^3$ can be cut by a plane $L$ in such a way that $F=D\cap L$ has a hyperbola as boundary curve. Since $F$ has asymptotes, it is not *qFW*, hence neither is the cone $D$.

The method of proof in implication (6) $\Longrightarrow$ (1) in Theorem 1 can be used to show that sub-level sets of convex polynomials are *qFW*-sets, see [3, Chap. II, §4, Thm. 13]. We obtain the following extension of [4, Thm. 3]:

**Corollary 3.** *Let $F_0$ be a qFW-set and let $f_1,\dots,f_m$ be convex polynomials on $F_0$ such that the set $F=\{x\in F_0: f_i(x)\le 0, i=1,\dots,m\}$ is non-empty. Let $f$ be a polynomial which is bounded below on $F$ and has at least one nonempty convex sub-level set on $F$. Then $f$ attains its infimum on $F$.* ☐

**Remark 9.** From Corollary 2 and Proposition 6 we learn that the class of *qFW*-sets is closed under finite intersections and cross products, while example 1 tells us that this is no longer true for *FW*-sets. Yet another invariance property of *qFW*-sets is the following:

**Corollary 4.** *Let $T:\mathbb{R}^n\to\mathbb{R}^m$ be an affine operator, and let $F\subset\mathbb{R}^m$ be a qFW-set. If $T^{-1}(F)$ is nonempty, then it is a qFW-set, too.*

*Proof.* We use property (4) of Theorem 1. Suppose $T^{-1}(F)$ had an $f$-asymptote $M$, then $T(M)$ would be an $f$-asymptote of $F$. ☐

**Corollary 5.** (See [4], [16, Cor. 27.3.1]). *Let $f$ be a polynomial which is convex and bounded below on a qFW-set $F$. Then $f$ attains its infimum on $F$.* □

The following consequence of Theorem 1 is surprising.

**Corollary 6.** *Let $F$ be a convex cone. Then the following are equivalent:*

(1) *$F$ is a FW-set;*

(2) *$F$ is a qFW-set;*

(3) *$F$ is polyhedral.*

*Proof.* (1) $\Longrightarrow$ (2) is clear, because $F$ is convex. (2) $\Longrightarrow$ (3): Let $F \subset \mathbb{R}^n$ be *qFW*, then by condition (iv) of Theorem 1 every orthogonal projection $P(F)$ on any two-dimensional subspace of $\mathbb{R}^n$ is closed. Therefore, by Mirkil's theorem, which we give as Lemma 1 below, $F$ is polyhedral. (3) $\Longrightarrow$ (1): By the classical Frank-and-Wolfe theorem every polyhedral convex cone is *FW*. □

**Lemma 1.** (Mirkil's theorem [15]). *Let $D$ be a convex cone in $\mathbb{R}^n$ such that every orthogonal projection on any two-dimensional subspace is closed. Then $D$ is polyhedral.* □

**Remark 10.** This result puts an end to hopes to get new results for the linear complementarity problem by investigating $FW$-cones.

We end this section with a nice consequence of Mirkil's theorem. First we need the following characterization of $f$-asymptotes:

**Proposition 7.** *For a closed convex set $F$ and a linear subspace $L$, the following statements are equivalent:*
1) *No translate of $L$ is an f-asymptote of $F$.*
2) *The orthogonal projection of $F$ onto the orthogonal complement $L^{\perp}$ is closed.*
3) *$F+L$ is closed.*

*Proof.* 1) $\Rightarrow$ 2). Let $x \in \overline{P_{L^{\perp}}(F)}$. Since $P_{L^{\perp}}^{-1}(x) = x+L$, we can easily prove that $\operatorname{dist}(F, x+L) = \operatorname{dist}(P_{L^{\perp}}(F), x) = 0$. Since $x+L$ is not an $f$-asymptote of $F$, we have $F \cap (x+L) \neq \emptyset$, which amounts to saying that $x \in P_{L^{\perp}}(F)$.

2) $\Rightarrow$ 3). Let $x_k \in F$ and $y_k \in L$ $(k = 1, 2, \ldots)$ be such that the sequence $x_k + y_k$ converges to some point $z$. Then $P_{L^{\perp}}(z) = \lim P_{L^{\perp}}(x_k + y_k) = \lim P_{L^{\perp}}(x_k) \in P_{L^{\perp}}(F)$ due to closedness of $P_{L^{\perp}}(F)$. But $P_{L^{\perp}}(F) = (F+L) \cap L^{\perp} \subset F+L$, hence $P_{L^{\perp}}(z) \in F+L$. Now $z = P_{L^{\perp}}(z) + P_L(z) \in F+L+L = F+L$.

3) $\Rightarrow$ 1). Let as assume that $x+L$ is an $f$-asymptote of $F$ for some $x$. Then $0 \leq \operatorname{dist}(x, F+L) \leq \operatorname{dist}\left(x, (F+L) \cap L^{\perp}\right) = \operatorname{dist}(x, P_{L^{\perp}}(F)) = \operatorname{dist}(F, x+L) = 0$, hence $\operatorname{dist}(x, F+L) = 0$. Since $F+L$ is closed, this implies $x \in F+L$. This is equivalent to saying that $F \cap (x+L) \neq \emptyset$, a contradiction to the assumption that $x+L$ is an $f$-asymptote of $F$. □

The consequence of Mirkil's Theorem we have in mind is the following:

**Proposition 8.** *For a closed convex cone $D$ in $\mathbb{R}^n$ (with $n > 2$), the following statements are equivalent:*
1) *$D$ is polyhedral.*
2) *$C+D$ is a convex polyhedron for every convex polyhedron $C$.*
3) *$L+D$ is closed for every $(n-2)$-dimensional subspace $L$.*
4) *$D$ has no $(n-2)$-dimensional f-asymptotes.*

*Proof.* Implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are immediate. Implication 3) $\Longrightarrow$ 1) is a consequence of 3) $\Rightarrow$ 2) of Proposition 7 combined with Mirkil's Theorem. Implication 3) $\Longrightarrow$ 4) follows from 3) $\Longrightarrow$ 1) of Proposition 7. Finally, implication 4) $\Longrightarrow$ 3) can be easily derived from implication 1) $\Longrightarrow$ 3) of Proposition 7. □

## 4. Motzkin type sets

Following [9, 10], a convex set $F$ is called Motzkin decomposable, if it may be written as the Minkowski sum of a compact convex set $C$ and a closed convex cone $D$, that is, $F = C + D$. Motzkin's classical result states that every convex polyhedron has such a decomposition. We extend this definition as follows:

**Definition 4.** A closed set $F \subset \mathbb{R}^n$ is called a *Motzkin set*, for short an *M*-set, if it can be written as $F = K + D$, where $K$ is a compact set, and $D$ is a closed convex cone.

We shall continue to reserve the term Motzkin decomposable for the case where the set $F$ is convex. A Motzkin set $F$ which is convex is then clearly Motzkin decomposable.

**Remark 11.** Let $F = K + D$ be a Motzkin set, then similarly to the convex case $D$ is uniquely determined by $F$. Indeed, taking convex hulls, we have $\text{co}(F) = \text{co}(K) + \text{co}(D) = \text{co}(K) + D$, hence $\text{co}(F)$ is a convex Motzkin set, i.e., a Motzkin decomposable set. Then from known results on Motzkin decomposable sets [9, 10], $D = 0^+\text{co}(F)$, the recession cone of $\text{co}(F)$. Now if we define the recession cone of $F$ in the same way as in the convex case, i.e., $0^+F = \{u \in \mathbb{R}^n : x + tu \in F \text{ for all } x \in F \text{ and all } t \geq 0\}$, then $0^+F \subset 0^+\text{co}(F) = D \subset 0^+F$, proving $D = 0^+F$. In particular, $F$ and $\text{co}(F)$ have the same recession cone.

**Theorem 2.** *Let $F$ be a Motzkin set in $\mathbb{R}^n$, represented as $F = K + D = K + 0^+F$. Then the following are equivalent:*

(1) *$F$ is a $FW$-set.*

(2) *The recession cone $0^+F$ of $F$ is polyhedral.*

(3) *$F$ has no $f$-asymptotes.*

*Proof.* We prove (1) $\Longrightarrow$ (2). Let $P$ be an orthogonal projection of $\mathbb{R}^n$ onto a subspace $L$ of $\mathbb{R}^n$. Since $F = K + D$ is a $FW$-set, $P(F)$ is closed. Since $P(F) = P(K) + P(D)$ and $\overline{P(F)} = P(K) + \overline{P(D)}$, this means $P(K) + P(D) = P(K) + \overline{P(D)}$. We have to show that this implies $P(D) = \overline{P(D)}$. This follows from the so-called *order cancellation law*, which we give as Lemma 2 below. It is applied to the convex sets $A = \overline{P(D)}$, $B = P(D)$, and for the compact set $P(K)$. This shows indeed $\overline{P(D)} = P(D)$. This means every projection of $D$ is closed, hence by Mirkil's theorem (Lemma 1), the cone $D$ is polyhedral.

**Lemma 2.** (Order cancellation law, see [10]). *Let $A, B \subset \mathbb{R}^n$ be convex sets, $K \subset \mathbb{R}^n$ a compact set. If $A + K \subset B + K$, then $A \subset B$.* □

Let us now prove (2) $\Longrightarrow$ (1). Write $F = K + D$ for $K$ compact and $D$ a polyhedral convex cone. Now consider a quadratic function $q(x) = \frac{1}{2}x^\top Ax + b^\top x$ bounded below by $\gamma$ on $F$. Hence

$$\inf_{x\in F} q(x) = \inf_{y\in K} \inf_{z\in D} q(y+z) = \inf_{y\in K} \left( q(y) + \inf_{z\in D} y^\top Az + q(z) \right) \geq \gamma. \tag{2}$$

Observe that for fixed $y \in K$ the function $q_y : z \mapsto y^\mathsf{T} Az + q(z)$ is bounded below on $D$ by $\eta = \gamma - \max_{y' \in C} q(y')$. Indeed, for $z \in D$ we have

$$\begin{aligned} y^\mathsf{T} Az + q(z) &\geq \left( q(y) + \inf_{z' \in D} y^\mathsf{T} Az' + q(z') \right) - q(y) \\ &\geq \inf_{y \in K} \left( q(y) + \inf_{z' \in D} y^\mathsf{T} Az' + q(z') \right) - \max_{y' \in K} q(y') \\ &\geq \gamma - \max_{y' \in K} q(y') = \eta. \end{aligned}$$

Since $q_y$ is a quadratic function bounded below on the polyhedral cone $D$, the inner infimum is attained at some $z = z(y)$. This is in fact the classical Frank and Wolfe theorem on a polyhedral cone. In consequence the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ defined as

$$f(y) = \inf_{z \in D} y^\mathsf{T} Az + q(z),$$

satisfies $f(y) = y^\mathsf{T} Az(y) + q(z(y)) > -\infty$ for every $y \in K$, so the compact set $K$ is contained in the domain of $f$. But now a stronger result holds, which one could call a parametric Frank and Wolfe theorem, and which we shall prove in Lemma 3 below. We show that $f$ is continuous relative to its domain. Once this is proved, the infimum (2) can then be written as

$$\inf_{x \in F} q(x) = \inf_{y \in K} q(y) + f(y),$$

and this is now attained by the Weierstrass extreme value theorem due to the continuity of $q + f$ on the compact $K$. Continuity of $f$ on $K$ is now a consequence of the following □

**Lemma 3.** *Let $D$ be a polyhedral convex cone and define*

$$f(c) = \inf_{x \in D} c^\mathsf{T} x + \tfrac{1}{2} x^\mathsf{T} Gx,$$

*where $G = G^\mathsf{T}$. Then* $\text{dom}(f)$ *is a polyhedral convex cone, and $f$ is continuous relative to* $\text{dom}(f)$.

*Proof.* If $x^\mathsf{T} Gx < 0$ for some $x \in D$, then $\text{dom}(f) = \emptyset$, so we may assume for the remainder of the proof that $x^\mathsf{T} Gx \geq 0$ for every $x \in D$. The proof is now divided into three parts. In part 1) we establish a formula for the domain $\text{dom}(f)$. In part 2) we use this formula to show that $\text{dom}(f)$ is polyhedral, and in part 3) we show that the latter implies continuity of $f$ relative to $\text{dom}(f)$.

1) We start by proving that

$$\text{dom}(f) = \left\{ c : c^\mathsf{T} x \geq 0 \text{ for every } x \in D \text{ such that } x^\mathsf{T} Gx = 0 \right\}. \tag{3}$$

The inclusion $\subseteq$ being obvious, we have to prove the following implication:

$$c^\mathsf{T} x \geq 0 \text{ for every } x \in D \text{ such that } x^\mathsf{T} Gx = 0 \Longrightarrow \inf_{x \in D} c^\mathsf{T} x + \tfrac{1}{2} x^\mathsf{T} Gx > -\infty.$$

We establish this by induction on the number $l$ of generators of $D$. The case $l = 1$ being clear, let $l > 1$, and suppose the implication is correct for every polyhedral convex cone $D'$ with $l' < l$

generators. Let $c$ be such that $c^\top x \geq 0$ for every $x \in D$ having $x^\top Gx = 0$. We have to show that $c \in \text{dom}(f)$. Assume on the contrary that

$$\inf_{x\in D} c^\top x + \tfrac{1}{2}x^\top Gx = -\infty, \tag{4}$$

and choose a sequence $x_k \in D$ with $\|x_k\| \to \infty$ such that

$$c^\top x_k + \tfrac{1}{2}x_k^\top Gx_k \longrightarrow -\infty. \tag{5}$$

Passing to a subsequence, we can assume that the sequence $y_k = x_k/\|x_k\|$ converges to some $y \in D$. We must have $y^\top Gy = 0$, as otherwise we would have $c^\top x_k + \frac{1}{2}x_k^\top Gx_k = \|x_k\|\, c^\top y_k + \frac{1}{2}\,\|x_k\|^2\, y_k^\top Gy_k \longrightarrow +\infty$, a contradiction. Hence, by our assumption, $c^\top y \geq 0$. We cannot have $c^\top y > 0$, as otherwise for large enough $k$ we would have $c^\top x_k = \|x_k\|\, c^\top y_k > 0$ and thus $c^\top x_k + \frac{1}{2}x_k^\top Gx_k > 0$ due to $x_k^\top Gx_k \geq 0$, which is impossible because of (5). Therefore $c^\top y = 0$. This will be used later.

Collecting more facts about $y$, note that as a consequence of our standing assumption $x^\top Gx \geq 0$ for $x \in D$, $y$ is a minimizer of the quadratic form $\frac{1}{2}x^\top Gx$ over $D$, which implies that $Gy$ belongs to the positive polar cone of $D$, that is, $x^\top Gy \geq 0$ for every $x \in D$. This property will also be used below.

Let $E = \{e_1, ..., e_l\}$ be the set of generating rays of $D$, and for $i = 1, ..., l$ denote by $D_i$ and $\widehat{D}_i$ the cones generated by $E \setminus \{e_i\}$ and $(E \setminus \{e_i\}) \cup \{y\}$, respectively. As the induction hypothesis applies to each $D_i$, we have $\inf_{x\in D_i} c^\top x + \frac{1}{2}x^\top Gx > -\infty$ for every $i$, so the infimum $m$ of $c^\top x + \frac{1}{2}x^\top Gx$ over $\bigcup_{i=1}^{l} D_i$ is finite.

Now observe that

$$D = \bigcup_{i=1}^{l} \widehat{D}_i. \tag{6}$$

Indeed, the inclusion $\supseteq$ being clear, take $x \in D$ and write it as $x = \sum_{i=1}^{l} \lambda_i e^i$ for certain $\lambda_i \geq 0$. Since $y \in D \setminus \{0\}$, we have $y = \sum_{i\in I} \mu_i e^i$ for some $\emptyset \neq I \subset \{1, \ldots, l\}$ and $\mu_i > 0$. Put $\nu = \min\{\lambda_i/\mu_i : i \in I\} =: \lambda_{i_0}/\mu_{i_0}$, then

$$x = \sum_{i\in I} \lambda_i e^i + \sum_{j\notin I} \lambda_j e^j + \nu \left( y - \sum_{i\in I} \mu_i e^i \right) = \sum_{i\in I} (\lambda_i - \nu\mu_i)\, e^i + \sum_{j\notin I} \lambda_j e^j + \nu y.$$

Since $\lambda_i - \nu\mu_i \geq 0$ for every $i \in I$, and $\lambda_{i_0} - \nu\mu_{i_0} = 0$, we have shown $x \in \widehat{D}_{i_0}$. That proves (6).

Now, using (6), for every $x \in D$ there exist $i \in \{1, ..., l\}$, $z \in D_i$, and $\lambda \geq 0$ such that $x = z + \lambda y$. We then have $c^\top x + \frac{1}{2}x^\top Gx = c^\top z + \lambda c^\top y + \frac{1}{2}z^\top Gz + \lambda z^\top Gy + \frac{1}{2}\lambda^2 y^\top Gy = c^\top z + \frac{1}{2}z^\top Gz + \lambda z^\top Gy \geq c^\top z + \frac{1}{2}z^\top Gz \geq m$, which gives $\inf_{x\in D} c^\top x + \frac{1}{2}x^\top Gx = m$, contradicting (4). This shows that our claim (3) was correct.

2) Now by the Farkas-Minkowski-Weyl theorem (cf. [16, Thm. 19.1] or [17, Cor. 7.1a]) the polyhedral cone $D$ is the linear image of the positive orthant of a space $\mathbb{R}^p$ of appropriate dimension, i.e. $D = \{Zu : u \in \mathbb{R}^p, u \geq 0\}$. Using (3), this implies

$$\text{dom}(f) = \{c : c^\top Zu \geq 0 \text{ for every } u \geq 0 \text{ such that } u^\top Z^\top GZu = 0\}.$$

Now observe that if $u \geq 0$ satisfies $u^\top Z^\top GZu = 0$, then it is a minimizer of the quadratic function $u^\top Z^\top GZu$ on the cone $u \geq 0$, hence $Z^\top GZu \geq 0$ by the Kuhn-Tucker conditions. Therefore we can write the set $P = \{u \in \mathbb{R}^p : u \geq 0, u^\top Z^\top GZu = 0\}$ as

$$P = \bigcup_{I \subset \{1,\dots,p\}} P_I,$$

where the $P_I$ are the polyhedral convex cones

$$P_I = \{u \geq 0 : Z^\top GZu \geq 0, u_i = 0 \text{ for all } i \in I, (Z^\top GZu)_j = 0 \text{ for all } j \notin I\}.$$

For every $I \subset \{1,\dots,p\}$ choose $m_I$ generators $u_{I1},\dots,u_{Im_I}$ of $P_I$. Then,

$$\begin{aligned} \operatorname{dom}(f) &= \left\{c : c^\top Zu \geq 0 \text{ for every } u \in P\right\} \\ &= \left\{c : c^\top Zu \geq 0 \text{ for every } u \in \textstyle\bigcup_{I \subset \{1,\dots,p\}} P_I\right\} \\ &= \textstyle\bigcap_{I \subset \{1,\dots,p\}} \left\{c : c^\top Zu \geq 0 \text{ for every } u \in P_I\right\} \\ &= \textstyle\bigcap_{I \subset \{1,\dots,p\}} \left\{c : c^\top Zu_{Ij} \geq 0 \text{ for all } j = 1,\dots,m_I\right\}. \end{aligned} \tag{7}$$

Since a finite intersection of polyhedral cones is polyhedral, this proves that $\operatorname{dom}(f)$ is a polyhedral convex cone.

3) To conclude, continuity of $f$ relative to its domain now follows from polyhedrality of $\operatorname{dom}(f)$, and using [16, Thm. 10.2], since $f$ is clearly concave and upper semicontinuous. This completes the proof of (2) $\implies$ (1).

(1) $\implies$ (3) was proved in Proposition 2. Let us prove (3) $\implies$ (2). By Mirkil's theorem (Lemma 1) it suffices to show that every orthogonal projection $P(F)$ is closed. Suppose this is not the case, and let $y \in \overline{P(F)} \setminus P(F)$. Let $L = y + \ker(P)$, then $F \cap L = \emptyset$. Now choose $y_n \in F$ such that $P(y_n) \to P(y) = y$. Then $y_n = P(y_n) + z_n$ with $z_n \in \ker(P)$. Hence $P(y) + z_n \in L$, but $\|(P(y_n) + z_n) - (P(y) + z_n)\| \to 0$, which shows $\operatorname{dist}(F, L) = 0$. That means $F$ has an $f$-asymptote, a contradiction. ☐

**Remark 12.** The main implication (2) $\implies$ (1) in Theorem 2 was first proved by Kummer [12]. Our proof of (2) $\implies$ (1) is slightly stronger in so far as it gives additional information on the polyhedrality of the domain of $f$ in Lemma 3.

**Remark 13.** We refer to Bank *et al.* [2, Thm. 5.5.1 (4)] for a result related to Lemma 3 in the case where $G \succeq 0$. For the indefinite case see also Tam [18].

**Remark 14.** The statement of Theorem 2 is no longer correct if one drops the hypothesis that $F$ is a Motzkin set. We take the convex $F = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, xy \geq 1\}$, then $F$, being limited by a hyperbola, has $f$-asymptotes, hence is not *qFW*, but $0^+F$ is the positive orthant, which is polyhedral.

**Corollary 7.** *A Motzkin decomposable set $F$ without $f$-asymptotes is Frank-and-Wolfe.*

*Proof.* Since $F$ has no $f$-asymptotes and is convex, it is a $qFW$-set by Theorem 1. But then by Theorem 2, $F$ is even a $FW$-set. ☐

## 5. Invariance properties of Motzkin *FW*-sets

We have seen in example 1 that intersections of $FW$-sets need no longer be $FW$-sets, not even when convexity is assumed. In contrast, the class of $qFW$-sets turned out closed under finite intersections. This rises the question whether more amenable sub-classes of the class of *FW*-sets with better invariance properties may be identified. In response we show in this chapter that the class of Motzkin *FW*-sets, for short *FWM*-sets, is better behaved with regard to invariance properties.

**Lemma 4.** *Consider a set of the form $K+D$, where $K$ is compact and $D$ is a polyhedral closed convex cone in $\mathbb{R}^n$. Let $L$ be a linear subspace of $\mathbb{R}^n$. Then there exists a compact set $K_0$ such that $(K+D)\cap L = K_0 + (D\cap L)$.*

*Proof.* 1) We assume for the time being that the cone $D\cap L$ is pointed. For fixed $x\in K$ consider the polyhedron $P_x := (x+D)\cap L$. Define $M(P_x) = \{x' \in P_x : (x' - (D\cap L))\cap P_x = \{x'\}\}$, and let $K(P_x)$ be the closed convex hull of $M(P_x)$. Then according to [9, Thm. 19] the set $K(P_x)$ is compact, and we have the minimal Motzkin decomposition $P_x = K(P_x) + (D\cap L)$. This uses the fact that $D\cap L$ is the recession cone of $P_x$. It follows that

$$(K+D)\cap L = \bigcup_{x\in K}(x+D)\cap L = \bigcup_{x\in K} K(P_x) + (D\cap L),$$

so all we have to do is show that the set $\cup_{x\in K}K(P_x)$ is bounded, as then its closure $K_0$ is the compact set announced in the statement of the Lemma. To prove boundedness of $\cup_{x\in K}K(P_x)$ it clearly suffices to show that $\cup_{x\in K}M(P_x)$ is bounded.

Let $\mathscr{F}$ be the finite set of faces of $D$, where we assume that $D$ itself is a face. Let $x' \in M(P_x)$, then $x'$ is in the relative interior of one of the faces $x+F$, $F\in\mathscr{F}$, of the shifted cone $x+D$.

We divide the faces $F\in\mathscr{F}$ of the cone $D$ into two types: $\mathscr{F}_1$ is the class of those faces $F\in\mathscr{F}$ for which there exists $d\in L$, $d\neq 0$, such that $d$ is a direction of recession of $F$, i.e., those where $F\cap L$ does not reduce to $\{0\}$. The class $\mathscr{F}_2$ gathers the remaining faces of $D$ which are not in the class $\mathscr{F}_1$.

Now suppose the set $\bigcup_{x\in K} M(P_x)$ is unbounded. Then there exists a sequence $x_k\in K$ and $x_k' \in M(P_{x_k})$ with $\|x_k'\|\to\infty$. From the above we know that each $x_k'$ is in the relative interior of $x_k + F_k$ for some $F_k\in\mathscr{F}$. Since there are only finitely many faces, we can extract a subsequence, also denoted $x_k$ and satisfying $\|x_k'\|\to\infty$, such that the $x_k'$ are relative interior points of $x_k+F$ for the same fixed face $F\in\mathscr{F}$. Due to compactness of $K$ we may, in addition, assume that $x_k\to x\in K$. Using the definition of $M(P_{x_k})$ write $x_k' = x_k + t_k d_k \in L$ with $d_k\in F\subset D$, $\|d_k\|=1$, $t_k>0$, $t_k\to\infty$. Passing to yet another subsequence, assume that $d_k\to d$, where $\|d\|=1$. It follows that $d\in L$, because in the expression $x_k'/t_k = x_k/t_k + d_k$ the middle term tends to 0 due to compactness of $K$ and $t_k\to\infty$, while the left hand term is in $L$ because $x_k'$ belongs to $L$. Since $F$ is a cone, it also follows that $x+\mathbb{R}_+d\subset x+F$, hence $d\in F$. This shows that the face $F$ is in the class $\mathscr{F}_1$.

2) So far we have shown that $\bigcup_{F\in\mathscr{F}_2}\{x'\in M(P_x) : x\in K, x'\in \mathrm{ri}(x+F)\}$ is a bounded set. It remains to prove that this set contains already all points $x'\in M(P_x)$, $x\in K$, i.e., that $x'\in M(P_x)$ cannot be a relative interior point of any of the faces $x+F$ with $F\in\mathscr{F}_1$.

3) Contrary to what is claimed, consider $x\in K\setminus L$ such that $x'\in M(P_x)$ satisfies $x'\in\mathrm{ri}(x+F)$ for some $F\in\mathscr{F}_1$. By definition of the class $\mathscr{F}_1$ there exists $d\in L\cap F$, $d\neq 0$. Since $x'\in L$ by the definition of $M(P_x)$, we have $x'+\mathbb{R}d\subset L$. But this line is also contained in $x+\mathrm{span}(F)$, because we have $d\in\mathrm{span}(F)$ and $x' = x+d'$ for some $d'\in F$, hence $x'+\mathbb{R}d\subset x+\mathrm{span}(F)$.

Since $x'$ is a relative interior point of $x+F$, there exists $\varepsilon>0$ such that $N_\varepsilon=\{x'+sd:|s|<\varepsilon\}$ is contained in $x+F$. Since $d\in F\cap L\subset D\cap L$, we have arrived at a contradiction with the fact that $x'\in M(P_x)$. Namely, moving in $N_\varepsilon$ we can stay in $P_x$ while going from $x'$ slightly in the direction of $-d\in-(D\cap L)$. This contradiction shows that what was claimed in 2) is true. The Lemma is therefore proved for pointed $D\cap L$.

4) Suppose now $D$ is allowed to contain lines. With a change of coordinates we may arrange that $\mathbb{R}^n=\mathbb{R}^m\times\mathbb{R}^p$ and $D\subset\mathbb{R}^m\times\{0\}$, where the possibility $p=0$ is not excluded and corresponds to the case where $D-D=\mathbb{R}^n$. Now consider the space $\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^p$ and define the cone $\widetilde{D}\subset\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^p$ as $\widetilde{D}=\{(x^+,x^-,0):x^\pm\in\mathbb{R}^m,x^\pm\geq 0,x^+-x^-\in D\}$. Then $\widetilde{D}$ is polyhedral and pointed. Let $T$ be the mapping $(x^+,x^-,y)\mapsto(x^+-x^-,y)$, then $T(\widetilde{D})=D$. Since $T$ maps $\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^p$ onto $\mathbb{R}^m\times\mathbb{R}^p$, there exists a compact set $\widetilde{K}\subset\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^p$ such that $T(\widetilde{K})=K$. Put $\widetilde{L}=T^{-1}(L)$. Now since $\widetilde{D}$ is pointed, the first part of the proof gives a compact $\widetilde{K}_0\subset\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^p$ such that $(\widetilde{K}+\widetilde{D})\cap\widetilde{L}=\widetilde{K}_0+(\widetilde{D}\cap\widetilde{L})$. Applying $T$ on both sides, and using the fact that $\widetilde{L}$ is a pre-image, we deduce $(K+D)\cap L=T(\widetilde{K}_0)+(D\cap L)$. On putting $K_0=T(\widetilde{K}_0)$ which is compact, we get the desired statement $(K+D)\cap L=K_0+(D\cap L)$. That completes the proof of the Lemma. □

**Corollary 8.** *Any finite intersection of sets of the form $K+D$ with $K$ compact and $D$ a polyhedral convex cone is again a set of this form.*

*Proof.* It suffices to consider the case of two sets $F_i=K_i+D_i$ in $\mathbb{R}^n$, $i=1,2$, with compact $K_i$ and $D_i$ polyhedral convex cones. We build the set $F=F_1\times F_2$ in $\mathbb{R}^n\times\mathbb{R}^n$, which is of the same form, because trivially $(K_1+D_1)\times(K_2+D_2)=(K_1\times K_2)+(D_1\times D_2)$, and since the product of two polyhedral cones is a polyhedral cone.

Now by Lemma 4 the intersection of $F_1\times F_2$ with the diagonal $\Delta=\{(x,x):x\in\mathbb{R}^n\}$ is a set of the form $\mathscr{K}+\mathscr{D}$ with $\mathscr{K}$ compact and $\mathscr{D}$ a polyhedral convex cone, because the diagonal is a linear subspace. Finally, $F_1\cap F_2$ is the image of $\mathscr{K}+\mathscr{D}$ under the projection $p:(x,y)\to x$ onto the first coordinate, hence is of the form $p(\mathscr{K})+p(\mathscr{D})$, and since $p(\mathscr{D})$ is a polyhedral convex cone, we are done. □

We conclude with the following invariance property of the class $FWM$:

**Proposition 9.** *If the pre-image of a FWM-set under an affine mapping is nonempty, then it is a FWM-set.*

*Proof.* Let $T$ be an affine mapping and $F$ be a $FWM$-set such that $T^{-1}(F)\neq\emptyset$. Since translates of $FWM$-sets are $FWM$, there is no loss of generality in assuming that $T$ is linear. Then the restriction of $T$ to $\ker(T)^\perp$ is a bijection from $\ker(T)^\perp$ onto $R(T)$, and one has

$$T^{-1}(F)=\left(T_{|\ker(T)^\perp}\right)^{-1}(F\cap R(T))+\ker(T).$$

Since $R(T)$ is a subspace, hence a convex polyhedron, and $T^{-1}(F)\neq\emptyset$, the set $F\cap R(T)$ is $FWM$ by Corollary 8. Since $\left(T_{|\ker(T)^\perp}\right)^{-1}$ is an isomorphism from $R(T)$ onto $\ker(T)^\perp$, the set $\left(T_{|\ker(T)^\perp}\right)^{-1}(F\cap R(T))$ is $FWM$. Hence it suffices to observe that $\ker(T)$, being a subspace, is $FWM$, and that the class of $FWM$-sets is closed under taking sums. □

**Remark 15.** It is worth mentioning that in general the affine pre-image of a Motzkin decomposable set need not be Motzkin decomposable. To wit, consider the ice cream cone $F$ in $\mathbb{R}^3$ and the mapping $T : (x_1,x_2,x_3) \mapsto (1,x_2,x_3)$, then the linear function $x_3 - x_2$ does not attain its infimum on $T^{-1}(F)$, which proves that $T^{-1}(F)$ is not Motzkin decomposable.

**Remark 16.** In Proposition 5 we had proved that the affine pre-image $T^{-1}(F)$ of a $FW$-set is $FW$ if $F$ is contained in the range of $T$. A priori this additional range condition cannot be removed, because we have no result which guarantees that $F \cap \text{range}(T)$ is still a $FW$-set (if nonempty). As we just saw, this range condition *can* be removed for $FWM$-sets, and also for $qFW$-sets, so these two classes are invariant under affine pre-images without further range restriction.

**Open question:** Let $F$ be a *FW*-set and $L$ a linear subspace, is $F \cap L$ a *FW*-set?

**Remark 17.** Altogether we have found the class of *FWM*-sets to be closed under finite products, finite intersections, images and pre-images under affine maps. If we call a set *FWMU* if it is a finite union of *FWM*-sets, then sets in this class are still *FW*-sets. By De Morgan's law the class *FWMU* remains closed under finite intersections. The class *FWMU* remains also closed under affine pre-images, because the pre-image of a union coincides with the union of the pre-images. Similarly the class $FWMU$ remains closed under affine images.

## 6. Parabolic sets

As we have seen in Theorem 2, the search for new *FW*-sets does not lead very far beyond polyhedrality within the Motzkin class, because if a Motzkin set $F = K + D$ is to be $FW$, then its recession cone $D = 0^+F$ must already be polyhedral. The question is therefore whether one can find $FW$-sets which exhibit non-polyhedral asymptotic behavior, those then being necessarily outside the Motzkin class. The following result shows that such *FW*-sets do indeed exist.

**Theorem 3.** (Luo and Zhang [13]). *Let $P$ be a closed convex polyhedron and define $F = \{x \in P : x^\mathsf{T} Qx + q^\top x + c \le 0\}$, where $Q = Q^\mathsf{T} \succeq 0$. Then $F$ is a $FW$-set.* □

The result generalizes the Frank and Wolfe theorem in the following sense: if we add just one convex quadratic constraint $x^\mathsf{T} Qx + q^\top x + c \le 0$ to a linearly constrained quadratic program, then finite infima of quadratics are still attained. As example 1 shows, adding a second convex quadratic constraint already fails.

The question is now can the Luo-Zhang theorem, just like the Frank-and-Wolf theorem, be extended from polyhedra $P$ to *FWM*-sets $F = K + D$? That means, if $F = K + D$ is a *FWM*-set, and if $Q = Q^\mathsf{T} \succeq 0$, will the set $\mathscr{F} = \{x \in F : x^\mathsf{T} Qx + q^\mathsf{T} x + c \le 0\}$ still be a *FW*-set ? We show by way of a counterexample that the answer is in the negative.

**Example 3.** We consider the cylinder $F = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : (x_1 - 1)^2 + x_2^2 \le 1\}$. Note that $F$ is a *FWM*-set, because it can be represented as $F = K + L$ for the compact convex set $K = \{(x_1,x_2,0,0) \in \mathbb{R}^4 : (x_1 - 1)^2 + x_2^2 \le 1\}$ and the subspace $L = \{0\} \times \{0\} \times \mathbb{R} \times \mathbb{R}$.

Now we add the convex quadratic constraint $x_3^2 \le x_4$ to the constraint set $F$, which leads to the set

$$\mathscr{F} = \{x \in F : x_3^2 \le x_4\} = \{x \in \mathbb{R}^4 : (x_1 - 1)^2 + x_2^2 \le 1,\ x_3^2 \le x_4\}.$$

We will show that $\mathscr{F}$ is no longer a *FW*-set. This means that the extension of Theorem 3 from polyhedra $P$ to *FWM*-sets $F$ fails.

Consider the quadratic function $q(x) = x_4x_1 - 2x_2x_3 + 2$. We claim that $q$ is bounded below on $\mathscr{F}$ by 0. Indeed, since $x_1 \geq 0$ on the feasible domain $\mathscr{F}$, we have $x_4x_1 \geq x_3^2x_1$ on the feasible domain, hence $q(x) \geq x_3^2x_1 - 2x_2x_3 + 2 = q(x_1,x_2,x_3,x_3^2)$, the expression on the right no longer depending on $x_4$. Let us compute the infimum of that expression on $\mathscr{F}$. This comes down to globally solving the program

$$(P) \quad \begin{array}{ll} \text{minimize} & x_3^2x_1 - 2x_2x_3 + 2 \\ \text{subject to} & (x_1-1)^2 + x_2^2 \leq 1 \end{array}$$

and it is not hard to see that $(P)$ has infimum 0, but that this infimum is not attained. (Solve for $x_3$ with fixed $x_1,x_2$ and show that the value at $(x_1,x_2,x_2/x_1)$ goes to 0 as $x_1 \to 0^+$, $(x_1-1)^2 + x_2^2 = 1$, but that 0 is not attained).

Now if $x^k \in \mathscr{F}$ is a minimizing sequence for $q$, then $\xi^k := (x_1^k, x_2^k, x_3^k, (x_3^k)^2) \in \mathscr{F}$ is also feasible and gives $q(x^k) \geq q(\xi^k)$, so the sequence $\xi^k$ is also minimizing, showing that the infimum of $q$ on $\mathscr{F}$ is the same as the infimum of $(P)$, which is zero. But then the infimum of $q$ on $\mathscr{F}$ could not be attained, as otherwise the infimum of $(P)$ would also be attained. Indeed, if the infimum of $q$ on $\mathscr{F}$ is attained at $\bar{x} \in \mathscr{F}$, then it must also be attained at $\bar{\xi} = (\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_3^2) \in \mathscr{F}$ because $q(\bar{x}) \geq q(\bar{\xi})$, and then the infimum of $(P)$ is attained at $(\bar{x}_1,\bar{x}_2,\bar{x}_3)$, contrary to what was shown.

**Remark 18.** We can write the set $\mathscr{F}$ as $\mathscr{F} = K' \times F'$, where $K' = \{(x_1,x_2) : (x_1-1)^2 + x_2^2 \leq 1\}$ is compact convex, and where $F'$ is the Luo-Zhang set $F' = \{(x_3,x_4) : x_3^2 \leq x_4\}$, which by Theorem 3 is a *FW*-set. This shows that the cross product of a convex *FW*-set (which is not *FWM*) and a compact convex set need no longer be a *FW*-set.

**Remark 19.** We can also write $\mathscr{F} = (K+L) \cap (F+M)$, where $L,M$ are linear subspaces of $\mathbb{R}^4$. Indeed, $K,L$ are as in Example 3, while $F = \{(0,0,x_3,x_4) : x_3^2 \leq x_4\}$ and $M = \mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\}$. Here $K+L$ is *FWM*, while $F+M$ is a *FW*-set by Theorem 3.

**Remark 20.** Note that $\mathscr{F}$ is a *qFW*-set by Proposition 6, see also [13, Cor. 2].

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