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Local convergence of an augmented Lagrangian method for matrix inequality constrained programming

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We consider nonlinear optimization programs with matrix inequality constraints, also known as nonlinear semidefinite programs. We prove local convergence for an augmented Lagrangian method which uses smooth spectral penalty functions. The sufficient second-order no-gap optimality condition and a suitable implicit function theorem are used to prove local linear convergence without the need to drive the penalty parameter to 0.

Keywords: Matrix inequality; Nonlinear semidefinite programming; Augmented Lagrangian; Spectral penalty; Implicit function theorem

1. Introduction

We consider mathematical optimization programs of the form

$$\begin{aligned} &\text{minimize} \quad f(x), \ x \in \mathbb{R}^n \\ &\text{subject to} \quad G(x) \preceq 0 \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, $G : \mathbb{R}^n \to \mathbb{S}^m$ a C^2 operator into the space $\mathbb{S}^m$ of $m \times m$ symmetric matrices, and where $\preceq 0$ means negative semidefinite. The constraint $G(x) \preceq 0$ is referred to as a matrix inequality, or as a nonlinear semidefinite constraint. We study augmented Lagrangian methods to solve (1) and develop a suitable local convergence theory.

Nonlinear programs (1) with matrix inequality constraints have been intensely studied since the 1990s. They arise in many applications in automatic control, finance and design engineering. Semidefinite programming (SDP) is a prominent special case of (1) which comes with a linear objective $f(x) = c^\top x$ and a linear matrix inequality $G(x) = A_0 + \sum_{i=1}^n A_i x_i \preceq 0$ in the constraint [1].

During the early 1990s, interior point methods were considered the only true way to solve (1), but other methods entered the scene from the late 1990s on, including nonsmooth and eigenvalue optimization [2–15] sequential SDP [16–19] and augmented Lagrangian methods.

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The use of augmented Lagrangians for (1) was proposed by Ben–Tal and Zibulevski in refs. [20,21]. Mosheyev and Zibulevski [22] studied several augmented Lagrangian models, and Kocvara and Stingl [23–25] developed the platforms PENNON and PENBMI to solve linear and bilinear semidefinite programs. Other approaches based on the augmented Lagrangian method are [26,27], [28–30] and [42–46]. In the present paper we obtain a local convergence theory for the methods [20,21,23–25].

The augmented Lagrangian models proposed in [20,21] are based on the idea of a spectral penalty function. Consider a convex C^2 function $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ with the following properties

(ϕ_1) ϕ is strictly convex, increasing and of class C^2 on dom(ϕ), which is open and contains $(-\infty, 0]$.
(ϕ_2) $\phi(0) = 0$.
(ϕ_3) $\phi'(0) = 1$.
(ϕ_4) $t\phi'(t) = \mathcal{O}(1)$ as $t \to -\infty$.
(ϕ_5) $t^2\phi''(t) = \mathcal{O}(1)$ as $t \to -\infty$.

Typical examples are

$$\phi(t) = \begin{cases} t + \frac{1}{2}t^2, & t \geq -\frac{1}{2} \\ -\frac{1}{4}\log(-2t) - \frac{3}{8}, & t \leq -\frac{1}{2} \end{cases} \quad \text{or} \quad \phi(t) = \begin{cases} \frac{1}{1-t} - 1, & t < 1 \\ +\infty, & \text{else} \end{cases} \tag{2}$$

Now define a matrix function $\Phi : \mathbb{S}^m \to \mathbb{S}^m$ associated with ϕ by setting

$$\Phi(X) = \Phi(Q \operatorname{diag} \lambda(X)\, Q^\top) = Q \operatorname{diag} \phi(\lambda(X))\, Q^\top, \tag{3}$$

where $X = Q \operatorname{diag} \lambda(X)\, Q^\top$ is a spectral decomposition of $X \in \mathbb{S}^m$, with $\lambda(X) \in \mathbb{R}^m$ the vector of eigenvalues of X in decreasing order, and where $\phi(\lambda) = (\phi(\lambda_1), \ldots, \phi(\lambda_m))$ for $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$. Observe that the operator Φ is independent of the choice of the orthonormal basis $Q(X) = [q_1(X), \ldots, q_m(X)]$ of eigenvectors of X, and may also be written as $\Phi(X) = \sum_{i=1}^m \phi(\lambda_i(X)) q_i(X) q_i(X)^\top$. Operators of this form are called *symmetric* and have been studied e.g. in [31,32]. Since $\phi(x) = x^n$ gives $\Phi(X) = X^n$, Φ is analytic for analytic ϕ. It can also be shown that Φ is of class C^2 whenever ϕ is of class C^2, see [33]. Given a penalty parameter $p > 0$ we define $\Phi_p(X) = p\,\Phi(p^{-1}X)$ and introduce the augmented Lagrangian function

$$F(x, U, p) = f(x) + U \bullet \Phi_p(G(x)), \tag{4}$$

where $U \in \mathbb{S}^m$ with $U \succeq 0$ is a Lagrange multiplier estimate. For fixed $U \succeq 0$ and $p > 0$ we now consider the unconstrained optimization program

$$\min_{x \in \mathbb{R}^n} F(x, U, p) \tag{5}$$

which we also call the *tangent program*. The augmented Lagrangian method is then defined as follows.

Augmented Lagrangian Algorithm

Fix $0 < \gamma < 1, 0 < \tau < 1$

1. Choose initial iterate x_1 and initial Lagrange multiplier estimate $U_1 \succeq 0$. Fix penalty $p_1 > 0$.
2. Given the current iterate x_k, Lagrange multiplier estimate $U_k \succeq 0$ and penalty $p_k > 0$, solve the tangent program
   $\min_{x\in\mathbb{R}^n} F(x, U_k, p_k)$
   possibly using x_k as a starting point for the inner iteration. The solution is x_{k+1}.
3. Update the Lagrange multiplier estimate by setting
   $U_{k+1} = \Phi'_{p_k}(G(x_{k+1}))U_k$
4. Update the penalty parameter by setting
   $$p_{k+1} = \begin{cases} p_k, & \text{if } \sigma(x_{k+1}, U_k, p_k) \le \tau\sigma(x_k, U_{k-1}, p_{k-1}) \\ \gamma p_k, & \text{else} \end{cases}$$
5. Increase counter k, and go back to step 2.

The mechanism is as follows. It is understood that solving the unconstrained program (5) is considerably easier than solving (1). We expect the sequence x_k to converge to a local minimum $\bar{x}$ of (1), while U_k converges to an associated Lagrange multiplier $\bar{U} \succeq 0$. The so-called first-order multiplier update rule $U_{k+1} = \Phi'_p(G(x_{k+1}))U_k$ in step 3 is used to improve the quality of the multiplier estimate before the next sweep. Axiom (ϕ_1) gives $\phi' > 0$, so that the operator Φ_p is strictly monotone, which means that $U_{k+1} \succeq 0$ as soon as $U_k \succeq 0$, and even $U_{k+1} \succ 0$ as soon as $U_k \succ 0$.

In step 4 the penalty parameter p_k is decreased when x_{k+1} does not make sufficient progress toward feasibility as compared to x_k. This progress is measured by a suitable primal-dual progress measure σ, defined as

$$\sigma(x^+, U, p) = \|U - \Phi'_p(G(x^+))U\| = \|U - U^+\|.$$

In fact, driving $p_k \to 0$ would ultimately force feasibility, but the rationale of the augmented Lagrangian scheme is that x_k may converge to $\bar{x}$ *without* forcing $p_k \to 0$. The objective of our local convergence analysis here is to show under what conditions this is possible, and that a linear rate of convergence can be obtained.

The matrix inequality constrained case has several challenges. Notice for instance that in contrast with the classical Hestenes–Powell–Rockafellar augmented Lagrangian [34–37], technical complications arise due to the fact that $(t, p) \mapsto p\phi(p^{-1}t)$ has a singularity at (0, 0). This difficulty leads to the concept of wedge convergence in section 7, Definitions 2 and 3, which plays a central role in our convergence analysis.

Yet another technical difficulty arises from the fact that we have to use the sufficient second-order no-gap optimality condition (11); cf. [38]. As we show by way of an example, it is *not* appropriate to use the *old* form of the second-order sufficient optimality condition (13) for matrix inequality constrained programs.

The structure of the paper is as follows. In sections 2–4 we recall useful facts from matrix constrained programming, covered essentially by [38]. Sections 5 and 7 prepare our case for the study of the analytic source function $\phi(t) = (1 - t)^{-1} - 1$. The main result is presented in section 6. Sections 8 and 9 are crucial and present technical results which combine the concept of wedge convergence with the second-order nogap optimality condition. The implicit function theorem is applied in section 10 under a special form given in Lemma 1. The central part of the proof, where the different threads are put together, is in section 11. We conclude with an example in section 12, showing that the nogap optimality condition is of the essence, and that the complications arising from it can *not* be avoided.

Our contribution is complementary to papers where global convergence proofs for augmented Lagrangians have been presented. For instance, [20] considers convergence of the present method in the convex case, [39] discusses and compares an even larger class of augmented Lagrangian models. Local convergence theory for the classical augmented Lagrangian method may be found in ref. [37], while local theory for classical programs based on smooth generating functions ϕ is presented in ref. [40].

Notation. The space of $m \times m$ symmetric matrices $\mathbb{S}^m$ is equipped with the scalar product $\mathrm{Tr}(XY) = X \bullet Y$. The negative cone in $\mathbb{S}^m$ is $\mathbb{S}^m_- = \{X \in \mathbb{S}^m : X \preceq 0\}$. For a symmetric expression $X = A + A^\top$ we shall sometimes write $X = A + *$ in order to facilitate the presentation. In the algorithm, x, U, p mean the current iterates, x^+, U^+, p^+ the next iterates, x^-, U^-, p^- those from the previous sweep. Notions from matrix inequality constrained mathematical programming are covered by [38].

2. First-order optimality condition

Let $\bar{x}$ be a local minimum of program (1) such that Robinson's constraint qualification [38, p. 72] is satisfied. Let $\bar{U} \succeq 0$ be a Lagrange multiplier associated with $\bar{x}$, then the Karush–Kuhn–Tucker conditions are

$$f'(\bar{x}) + G'(\bar{x})^\star \bar{U} = 0,\ G(\bar{x}) \preceq 0,\ \bar{U} \succeq 0,\ G(\bar{x}) \bullet \bar{U} = 0. \tag{6}$$

Here the adjoint operator $G'(x)^\star$ is defined as follows. Let $G_i(x) = \partial G(x)/\partial x_i \in \mathbb{S}^m$, $i = 1, \ldots, n$, then $G'(x)^\star Y = (G_1(x) \bullet Y, \ldots, G_n(x) \bullet Y) \in \mathbb{R}^n$, see [38,41].

As is well-known, complementarity $G(\bar{x}) \perp \bar{U}$ in (6) implies that $\bar{U}$ and $G(\bar{x})$ commute, and may therefore be diagonalized simultaneously. Assuming without loss that $G(\bar{x})$ and $\bar{U}$ are already diagonal matrices, we have

$$G(\bar{x}) = \mathrm{diag}[\bar{g}_1, \ldots, \bar{g}_s, 0_{m-s}], \quad \bar{U} = \mathrm{diag}[0_s, \bar{u}_{s+1}, \ldots, \bar{u}_m] \tag{7}$$

where $\bar{g}_i < 0$ and $\bar{u}_j \geq 0$. Strict complementarity is satisfied as soon as $\bar{u}_j > 0$ for $j = s+1, \ldots, m$.

3. Second-order optimality condition

Let us now consider the second order sufficient optimality condition as proposed in [38,41]. The Lagrangian of (1) is

$$L(x, U) = f(x) + U \bullet G(x). \tag{8}$$

Following [41 formula (37)], the critical cone at $(\bar{x}, \bar{U})$ is

$$C(\bar{x}) = \{h \in \mathbb{R}^n : \bar{U} \bullet [G'(\bar{x})h] = 0,\ G'(\bar{x})h \in T(\mathbb{S}^m_-, G(\bar{x}))\},$$

where $T(\mathbb{S}^m_-, G)$ is the tangent cone to $\mathbb{S}^m_-$ at $G \in \mathbb{S}^m_-$. This tangent cone is

$$T(\mathbb{S}^m_-, G) = \{Z \in \mathbb{S}^m : E^\top Z E \preceq 0\},$$

where E is a $m \times (m-s)$ matrix whose columns form an orthonormal basis of the null space of G; cf. [38, p. 474]. Due to (7), the null space of $\bar{G} = G(\bar{x})$ is spanned by the $m-s$ unit vectors $e_{s+1}, \dots, e_m$. That means, if we partition the matrix Z as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^\top & Z_{22} \end{bmatrix}, \quad Z_{11} \in \mathbb{S}^s, Z_{22} \in \mathbb{S}^{m-s}, \tag{9}$$

then $T(\mathbb{S}^m_-, \bar{G}) = \{Z \in \mathbb{S}^m : Z_{22} \preceq 0\}$. Therefore, the critical cone may be written as

$$C(\bar{x}) = \{h \in \mathbb{R}^n : \bar{U} \bullet [G'(\bar{x})h] = 0, [G'(\bar{x})h]_{22} \preceq 0\}.$$

Naturally, the first of these two conditions may also be written as

$$\bar{U} \bullet [G'(\bar{x})h] = \bar{U}_{22} \bullet [G'(\bar{x})h]_{22} = 0.$$

Strict complementarity, $\bar{u}_i > 0$ for $i = s+1, \dots, m$, in tandem with $[G'(\bar{x})h]_{22} \preceq 0$ implies $[G'(\bar{x})h]_{22} = 0$. In other words, under strict complementarity the critical cone is the linear subspace

$$C(\bar{x}) = \{h \in \mathbb{R}^n : [G'(\bar{x})h]_{22} = 0\}. \tag{10}$$

Let us now present the so-called no-gap second-order sufficient optimality condition. It reads

$$h^\top [L_{xx}(\bar{x}, \bar{U}) + \mathcal{H}(\bar{x}, \bar{U})]h > 0 \text{ for every } h \in C(\bar{x}), h \neq 0, \tag{11}$$

where $L_{xx}(\bar{x}, \bar{U})$ is the Hessian of the Lagrangian (8), and where $\mathcal{H}(\bar{x}, \bar{U})$ is a term reflecting curvature information of the feasible domain at $\bar{x}$. According to [41, formula (40)], this term is of the form

$$[\mathcal{H}(\bar{x}, \bar{U})]_{ij} = -2\bar{U} \bullet (G_i(\bar{x})[G(\bar{x})]^\dagger G_j(\bar{x})),$$

or in a compact notation

$$\mathcal{H}(\bar{x}, \bar{U}) = -2 \left(\frac{\partial G(\bar{x})}{\partial x}\right)^\top (\bar{U} \otimes [G(\bar{x})]^\dagger) \left(\frac{\partial G(\bar{x})}{\partial x}\right),$$

where $M^\dagger$ denotes the pseudo inverse of a matrix M, $\otimes$ the Kronecker product, and where $\partial G(x)/\partial x$ denotes the $n^2 \times m$ matrix $[\operatorname{vec} G_1(x), \dots, \operatorname{vec} G_m(x)]$. Consequently, we obtain for the curvature term

$$\begin{aligned} h^\top \mathcal{H}(\bar{x}, \bar{U})h &= \sum_{i,j=1}^n h_i h_j \left(-2\bar{U} \bullet G_i(\bar{x})G(\bar{x})^\dagger G_j(\bar{x})\right) \\ &= -2\bar{U} \bullet \left(\sum_{i,j=1}^n h_i h_j G_i(\bar{x})G(\bar{x})^\dagger G_j(\bar{x})\right) \\ &= -2\bar{U} \bullet \left(\sum_{i=1}^n h_i G_i(\bar{x})G(\bar{x})^\dagger \sum_{j=1}^n h_j G_j(\bar{x})\right) \\ &= -2\bar{U} \bullet [G'(\bar{x})h]G(\bar{x})^\dagger[G'(\bar{x})h]. \end{aligned}$$

Due to the special structure (7), (9), we may develop this expression further, which yields

$$h^\top \mathcal{H}(\bar{x}, \bar{U})h = -2\operatorname{diag}(\bar{u}_{s+1}, \dots, \bar{u}_m) \bullet [G'(\bar{x})h]_{12}^\top \operatorname{diag}\left(\frac{1}{\bar{g}_1}, \dots, \frac{1}{\bar{g}_s}\right)[G'(\bar{x})h]_{12}. \tag{12}$$

As can be seen, this term is ≥ 0, which means that the no-gap condition (11) is weaker than the 'classical' second-order sufficient condition:

$$h^\top L_{xx}(\bar{x}, \bar{U})h > 0 \ \text{ for all } \ h \in C(\bar{x}) \setminus \{0\}. \tag{13}$$

In fact, as we shall see in section 12, this condition, which is still used by many authors to extend results from classical nonlinear programming to matrix inequality constrained programming in a straightforward way, is too strong to be realistic. Results based on (13) are therefore of little interest.

4. Constraint qualification

We need one more element, a generalization of the linear independence constraint qualification (LICQ) from classical nonlinear programming. Let $\bar{G} = G(\bar{x})$ and let E be a $m \times (m - s)$ matrix whose $m - s$ columns form an orthonormal basis of the null space of $\bar{G}$, then we say that the generalized LICQ condition holds if

$$W \longmapsto (E^\top G_1(\bar{x})E \bullet W, \ldots, E^\top G_n(\bar{x})E \bullet W), \quad \mathbb{S}^{m-s} \longrightarrow \mathbb{R}^n \quad \text{is injective.} \tag{14}$$

In the situation (7), condition LICQ is equivalent to the following:

$$W \longmapsto (G_1(\bar{x})_{22} \bullet W, \ldots, G_n(\bar{x})_{22} \bullet W), \quad \mathbb{S}^{m-s} \longrightarrow \mathbb{R}^n \quad \text{is injective.} \tag{15}$$

As in classical nonlinear programming, LICQ implies uniqueness of the Lagrange multiplier $\bar{U}$. Notice that (15) appears fairly restrictive at first sight, because it requires in particular that $n \geq (m - s)(m - s + 1)/2$. However, as we will see, this condition reduces to the classical LICQ condition if the operator G is diagonal.

Indeed, suppose more generally that $G : \mathbb{R}^n \to \mathbb{S}^{m_1} \oplus \cdots \oplus \mathbb{S}^{m_b} \subset \mathbb{S}^m$ has a block diagonal structure with b blocks, where $m_1 + \cdots + m_b = m$. Then multipliers U and partial derivatives $G_j(x)$ have the same structure, and the linear independence condition can be restricted to that space, i.e., (15) is required injective on the space of $W \in \mathbb{S}^{m-s}$ with this structure. In particular, this means $n \geq \sum_{j=1}^{b}(m_j - s_j)(m_j - s_j + 1)/2$, where in each block j, we assume that s_j eigenvalues are < 0, the remaining $m_j - s_j$ eigenvalues are active at 0.

In the special case where $G(x)$ is diagonal, we have $m_j = 1$ and $m = b$. Assuming that p constraints are active, we would have $s_1 = \cdots = s_p = 0$, $s_{p+1} = \cdots = s_m = 1$. Here the LICQ condition coincides with the classical one, and the dimension condition simply reduces to $\sum_{j=1}^{m}(m_j - s_j)(m_j - s_j + 1)/2 = p \leq n$, which is of course necessary if the p active constraint gradients are to be linearly independent at $\bar{x}$.

5. Analytic source function $\phi(t) = (1 - t)^{-1} - 1$

In this section we will start analyzing the augmented Lagrangian model in the special case of the source function $\phi(t) = (1 - t)^{-1} - 1$, which was proposed in [21,22] and later used to develop the software tool PENNON [23,24]. In this case, analyticity of ϕ allows explicit computations of the derivatives of the associated Φ. Starting out with $\Phi(X) = (I - X)^{-1} - I$, we consider

$$\Phi_p(G(x)) := p\,\Phi(p^{-1}G(x)) = p(I - p^{-1}G(x))^{-1} - pI.$$

Expanding the C^2 operator $G(x+d) = G(x) + G'(x)d + (1/2)G''(x)[d,d] + \mathcal{O}(||d||^3)$, we have

$$\begin{aligned}\Phi_p(G(x+d)) = \Phi_p(G(x)) &+ (I - p^{-1}G(x))^{-1}[G'(x)d](I - p^{-1}G(x))^{-1}\\ &+ (I - p^{-1}G(x))^{-1}\left[\frac{1}{2}G''(x)[d,d]\right](I - p^{-1}G(x))^{-1}\\ &+ p^{-1}\,(I - p^{-1}G(x))^{-1}[G'(x)d](I - p^{-1}G(x))^{-1}\\ &\times [G'(x)d](I - p^{-1}G(x))^{-1} + \mathcal{O}(\|d\|^3).\end{aligned}$$

Therefore, the expansion of the penalty term $U \bullet \Phi_p(G(x))$ in (4) is

$$\begin{aligned}U \bullet \Phi_p(G(x+d)) = U \bullet \Phi_p(G(x)) &+ [G'(x)d] \bullet U^+(x,U,p)\\ &+ \frac{1}{2}\left\{[G''(x)[d,d]] + 2p^{-1}[G'(x)d]\right.\\ &\times \left.(I - p^{-1}G(x))^{-1}[G'(x)d]\right\} \bullet U^+(x,U,p) + \mathcal{O}(\|d\|^3),\end{aligned}$$

where we put $U^+(x,U,p) := (I - p^{-1}G(x))^{-1}U(I - p^{-1}G(x))^{-1}$. Using the standard notations

$$G_i(x) = \frac{\partial G(x)}{\partial x_i} \in \mathbb{S}^m, \quad G_{ij}(x) = \frac{\partial^2 G(x)}{\partial x_i \partial x_j} \in \mathbb{S}^m,$$

we derive the following formulas:

$$\begin{aligned}F_x(x,U,p) &= f'(x) + G'(x)^\star U^+(x,U,p)\\ &= f'(x) + (G_1(x) \bullet U^+(x,U,p), \ldots, G_n(x) \bullet U^+(x,U,p))\end{aligned} \tag{16}$$

and

$$\begin{aligned}F_{xx}(x,U,p)_{ij} = f''(x)_{ij} + G_{ij}(x) \bullet U^+(x,U,p) &+ 2p^{-1}(G_i(x)(I - p^{-1}G(x))^{-1}G_j(x)\\ &+ G_j(x)(I - p^{-1}G(x))^{-1}G_i(x)) \bullet U^+(x,U,p).\end{aligned} \tag{17}$$

Notice that (16) gives the following formula

$$L_x(x, U^+(x,U,p)) = F_x(x,U,p) \tag{18}$$

whose analogue in the classical setting is well-known [37, p. 104ff]. It will be of use later.

The first-order update formula $U^+ = \Phi'_p(G(x^+))U$ takes the following explicit form

$$U^+ = (I - p^{-1}G(x^+))^{-1}U(I - p^{-1}G(x^+))^{-1}. \tag{19}$$

Finally, we will also make use of the partial derivative F_{xU}, which is readily obtained as

$$F_{xU}(x,U,p)\delta U = (G_i(x) \bullet ((I - p^{-1}G(x))^{-1}\delta U(I - p^{-1}G(x))^{-1}))_{i=1}^n \in \mathbb{R}^n. \tag{20}$$

Using the notation $Z_p(x) = (I - p^{-1}G(x))^{-1}$ and the definition of the adjoint operator $G'(x)^\star$, we can write this more compactly as

$$F_{xU}(x,U,p)\delta U = G'(x)^\star[Z_p(x)\delta U Z_p(x)].$$

6. Main theorem

Let $\bar{x}$ be a local minimum of (1) which is a KKT-point with unique associated Lagrange multiplier matrix $\bar{U}$. We consider the following hypotheses at $\bar{x}$:

(H_1) Strict complementarity (7).
(H_2) The second-order sufficient no-gap optimality condition (11).
(H_3) The generalized LICQ condition (15).

THEOREM. *Let $\bar{x}$ be a local minimum of* (1) *with associated Lagrange multiplier $\bar{U}$ such that hypotheses* (H_1)–(H_3) *are satisfied. Then there exists a neighborhood N of $\bar{U}$, a neighborhood $\mathcal{U}$ of $\bar{x}$, and $\overline{p} > 0$ such that the following conditions are satisfied*:

1. *Whenever $U_1 \in N$ and $\gamma\overline{p} < p_1 \leq \overline{p}$, then the sequences U_k, $p_k > 0$ and x_k generated by the augmented Lagrangian algorithm are well-defined if x_{k+1} is the local minimum of $\min_{x\in\mathbb{R}^n} F(x, U_k, p_k)$ in $\mathcal{U}$. The sequence U_k stays in N, and x_{k+1} is the unique critical point of* (5) *in $\mathcal{U}$.*
2. *The sequence U_k converges to $\bar{U}$ with Q-linear speed, and x_k converges to $\bar{x}$ with R-linear speed.*
3. *The sequence $p_k > 0$ is constant from some index k_1 on.*

The proof of this Theorem requires the preliminaries in sections 2–5, while the principal arguments are covered by sections 7–11.

7. Preparations

In this section we consider technical notions needed for our convergence proof.

LEMMA 1. *Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ and let $H : \Omega \to \mathbb{R}^n$ be of class $C^k(\Omega)$ for some $k \geq 1$. Let K^* be a compact subset of $\mathbb{R}^m$ and suppose there exists a vector $x^* \in \mathbb{R}^n$ with $\{x^*\} \times K^* \subset \Omega$ such that $H(x^*, y) = 0$ for every $y \in K^*$. Suppose $H_x(x^*, y)$ is invertible for every $y \in K^*$. Then there exists a neighborhood W of $\{x^*\} \times K^*$, a neighborhood V of K^*, and a function $x(\cdot) : V \to \mathbb{R}^n$ of class C^k such that $H(x(y), y) = 0$ for every $y \in V$ and $x(y) = x^*$ for every $y \in K^*$. The function is unique in the sense that for every $(x, y) \in W$, $H(x, y) = 0$ if and only if $y \in V$ and $x = x(y)$. Moreover,*

$$x'(y) = -[H_x(x(y), y)]^{-1} H_y(x(y), y).$$

This coincides with the usual implicit function theorem when the set $K^* = \{y^*\}$ is a singleton set.

The following technical notion will be helpful in our convergence proof.

DEFINITION 2. *The sequence $(x_k, U_k, p_k) \in \mathbb{R}^n \times \mathbb{S}^m \times \mathbb{R}$ is said to wedge-converge to $(\bar{x}, \bar{U}, 0)$, noted $(x_k, U_k, p_k) \xrightarrow{w} (\bar{x}, \bar{U}, 0)$ if $x_k \to \bar{x}$, $U_k \to \bar{U}$, $p_k \to 0$ in such a way that $(x_k - \bar{x})/p_k \to 0$, $(U_k - \bar{U})/p_k \to 0$. Similarly, (x_k, p_k) wedge converges to $(\bar{x}, 0)$ in $\mathbb{R}^n \times \mathbb{R}$ if $x_k \to \bar{x}$, and $p_k \to 0$ such that $(x_k - \bar{x})/p_k \to 0$.*

The following concept will also be useful. It represents a different way to describe wedge convergence.

DEFINITION 3. *The set*

$$\mathcal{W}(\epsilon) = \left\{(x, U, p) : \frac{\|x - \bar{x}\|}{p} \leq \epsilon, \frac{\|U - \bar{U}\|}{p} \leq \epsilon, 0 < p \leq \epsilon\right\}$$

is called a wedge neighborhood of $(\bar{x}, \bar{U}, 0)$. *Similarly, the set*

$$\mathcal{W}'(\epsilon) = \left\{(x, p) : \frac{\|x - \bar{x}\|}{p} \leq \epsilon, 0 < p \leq \epsilon\right\}$$

is a wedge neighborhood of $(\bar{x}, 0)$.

8. Lemmas with wedge convergence I

The results in this section exploit properties of the augmented Lagrangian function as it relates to wedge convergence. We can think of this part as collecting prior information, which will enable us later on (in section 11) to fix a parameter interval $\mathcal{I} = [\underline{p}, \overline{p}]$.

LEMMA 4. *Assume hypotheses* (H_1)–(H_3) *are satisfied. Then there exist* $\epsilon_1 > 0$ *and* $K_1 > 0$ *such that*

$$\|F_{xx}(x, U, p)^{-1}\| \leq K_1 \tag{21}$$

for every $(x, U, p) \in \mathcal{W}(\epsilon_1)$. *Equivalently, there exists* $\rho > 0$ *such that*

$$F_{xx}(x, U, p) \succeq \rho I \succ 0 \tag{22}$$

for every $(x, U, p) \in \mathcal{W}(\epsilon_1)$.

Proof. 1) It suffices to prove that $F_{xx}(x, U, p) \succeq \rho I \succ 0$ on a wedge neighborhood of $(\bar{x}, \bar{U}, 0)$. Assume on the contrary that there exist $x_k \to \bar{x}$, $U_k \to \bar{U}$, $p_k \to 0$ such that $(x_k - \bar{x})/p_k \to 0$, $(U_k - \bar{U})/p_k \to 0$ but $d_k^\top F_{xx}(x_k, U_k, p_k)d_k \leq \delta_k \to 0$ for certain unit vectors d_k. Passing to a subsequence if necessary, we may assume that $d_k \to d$ for a unit vector d, and that $d_k^\top F_{xx}(x_k, U_k, p_k)d_k$ converges to a quantity $-\vartheta$ with $\vartheta \geq 0$. A priori, we could have $\vartheta = +\infty$, but we will see in a moment that this possibility can be ruled out.

2) Now observe that with (17), writing $Z_k := Z_{p_k}(x_k) = (I - p_k^{-1}G(x_k))^{-1}$, we obtain

$$\begin{aligned} d_k^\top F_{xx}(x_k, U_k, p_k)d_k &= d_k^\top f''(x_k)d_k + U_k \bullet Z_k[G''(x_k)[d_k, d_k]]Z_k \\ &\quad + 2p_k^{-1}U_k \bullet Z_k[G'(x_k)d_k]Z_k[G'(x_k)d_k]Z_k \\ &= d_k^\top L_{xx}(x_k, Z_kU_kZ_k)d_k + 2p_k^{-1}U_k \bullet Z_k[G'(x_k)d_k]Z_k[G'(x_k)d_k]Z_k. \end{aligned} \tag{23}$$

Let us show that the term $d_k^\top L_{xx}(\cdots)d_k$ on the right-hand side of (23) converges to $d^\top L_{xx}(\bar{x}, \bar{U})d$. This follows as soon as we show that $Z_kU_kZ_k$ converges to $\bar{U}$. To prove this, consider a spectral decomposition $G(x_k) = Q_kG_kQ_k^\top$, where $G_k = \text{diag}(g_1^k, \ldots, g_m^k)$. Then $Z_k = (I - p_k^{-1}G(x_k))^{-1} = Q_kD_kQ_k^\top$, where the diagonal matrix $D_k = (I - p_k^{-1}G_k)^{-1}$ has diagonal entries $p_k/(p_k - g_i^k)$. Selecting a convergent subsequence $Q_k \to Q$, we have $Z_k = Q_kD_kQ_k^\top \to Q\text{diag}(0_s, I_{m-s})Q^\top$, because $(g_i^k - \bar{g}_i)/p_k \to 0$ by wedge convergence, and $\bar{g}_i = 0$ for $i = s + 1, \ldots, m$, while $\bar{g}_i < 0$ for $i = 1, \ldots, s$. Here Q is an orthogonal matrix which gives a spectral decomposition of $G(\bar{x})$. According to (7), this means

$G(\bar{x}) = Q\bar{G}Q^\top = \bar{G} = \text{diag}(\bar{g}_1, \dots, \bar{g}_m)$. Since $\bar{U}$ and $G(\bar{x})$ commute, Q also diagonalizes $\bar{U}$, i.e., with (7) we have $Q\bar{U}Q^\top = \bar{U}$. Therefore, with $\widetilde{U}_k = Q_k^\top U_k Q_k$ we have $Z_k U_k Z_k = Q_k(D_k \widetilde{U}_k D_k)Q_k^\top \to Q\text{diag}(0_s, I_{m-s})\bar{U}\text{diag}(0_s, I_{m-s})Q^\top = Q\bar{U}Q^\top = \bar{U}$ as claimed. The result being the same for any convergent subsequence $Q_k \to Q$, the conclusion is that $Z_k U_k Z_k \to \bar{U}$.

3) Let us now look at the second term on the right-hand side of (23), which is nonnegative. Since the first term $d_k^\top L_{xx}(\cdots)d_k$ on the right-hand side of (23) converges to $d^\top L_{xx}(\bar{x}, \bar{U})$ $d \in \mathbb{R}$, non-negativity of the second term on the right-hand side of (23) implies that the limit $-\vartheta$ of $d_k^\top F_{xx}(\cdots)d_k$ on the left-hand side of (23) must be finite. In consequence, the limit of the second term on the right-hand side of (23) is also finite and ≥ 0.

This term is of the form $2p_k^{-1}\Xi_k \geq 0$ with $\Xi_k = \Psi^k(I - p_k^{-1}G(x_k))^{-1}\Psi^k \bullet U^+(x_k, U_k, p_k)$, where we have for ease of notation put $\Psi^k := G'(x_k)d_k$. Since $2p_k^{-1}\Xi_k$ converges by what was seen above, and since $p_k^{-1} \to +\infty$, it follows that $\Xi_k \to 0$.

Now as we have seen, $U^+(x_k, U_k, p_k) = Z_k U_k Z_k \to \bar{U}$, and $\Psi^k Z_k \Psi^k \to \bar{\Psi}\bar{Z}\bar{\Psi} = \bar{\Psi}\text{diag}[0_s, I_{m-s}]\bar{\Psi}$, where $\bar{\Psi} := G'(\bar{x})d$, $\bar{Z} := \text{diag}[0_s, I_{m-s}]$. Therefore, we have $\Xi_k \to \bar{U}_{22} \bullet \bar{\Psi}_{22}\bar{\Psi}_{22} = 0$. Since $\bar{U}_{22} \succ 0$ by strict complementarity and $\bar{\Psi}_{22}\bar{\Psi}_{22} \succeq 0$, we deduce $\bar{\Psi}_{22} = [G'(\bar{x})d]_{22} = 0$. In other words, $d \in C(\bar{x})$ is a critical direction (10).

Using this information we get back to the convergent term $2p_k^{-1}\Xi_k$, which we write as

$$2p_k^{-1}\Xi_k = 2p_k^{-1}Z_k\Psi^k Z_k \Psi^k Z_k \bullet U_k.$$

As before let Q_k an orthogonal matrix which diagonalizes $G(x_k)$, $G(x_k) = Q_k G_k Q_k^\top$ with $G_k = \text{diag}(g_1^k, \dots, g_m^k)$. Then $Z_k = (I - p_k^{-1}G(x_k))^{-1} = Q_k(I - p_k^{-1}G_k)^{-1}Q_k^\top$. Let us introduce the matrices $\widetilde{\Psi}^k = Q_k^\top \Psi^k Q_k$ and $\widetilde{U}_k = Q_k^\top U_k Q_k$. We decompose according to (9):

$$\widetilde{\Psi}^k = \begin{bmatrix} \widetilde{\Psi}_{11}^k & \widetilde{\Psi}_{12}^k \\ \widetilde{\Psi}_{12}^{k\top} & \widetilde{\Psi}_{22}^k \end{bmatrix}, \quad \widetilde{U}_k = \begin{bmatrix} \widetilde{U}_{11}^k & \widetilde{U}_{12}^k \\ \widetilde{U}_{12}^{k\top} & \widetilde{U}_{22}^k \end{bmatrix}$$

and expand the term $2p_k^{-1}\Xi_k$ as follows:

$$\begin{aligned} 2p_k^{-1}\Xi_k &= 2p_k^{-1}(I - p_k^{-1}G_k)^{-1}\widetilde{\Psi}^k(I - p_k^{-1}G_k)^{-1}\widetilde{\Psi}^k(I - p_k^{-1}G_k)^{-1} \bullet \widetilde{U}_k \\ &= 2p_k^{-1}\widetilde{U}_{11}^k \bullet ((I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{11}^k(I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{11}^k(I - p_k^{-1}G_k)_{11}^{-1} \\ &\qquad + (I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{12}^k(I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{12}^{k\top}(I - p_k^{-1}G_k)_{11}^{-1}) \\ &\quad + 2p_k^{-1}\widetilde{U}_{12}^k \bullet ((I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{12}^{k\top}(I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{11}^k(I - p_k^{-1}G_k)_{11}^{-1} \\ &\qquad + (I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{12}^{k\top}(I - p_k^{-1}G_k)_{11}^{-1}) \\ &\quad + 2p_k^{-1}\widetilde{U}_{12}^{k\top} \bullet ((I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{11}^k(I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{12}^k(I - p_k^{-1}G_k)_{22}^{-1} \\ &\qquad + (I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{12}^k(I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1}) \\ &\quad + 2p_k^{-1}\widetilde{U}_{22}^k \bullet ((I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{12}^{k\top}(I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{12}^k(I - p_k^{-1}G_k)_{22}^{-1} \\ &\qquad + (I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1}). \end{aligned}$$

Now observe that $\widetilde{U}_k \xrightarrow{w} \bar{U}$, so that $2p_k^{-1}\widetilde{U}_{11}^k \to 0$ and $2p_k^{-1}\widetilde{U}_{12}^k \to 0$. Therefore, the first three terms of the above expression $2p_k^{-1}\widetilde{U}_{11}^k \bullet (\cdots)$, $2p_k^{-1}\widetilde{U}_{12}^k \bullet (\cdots)$ and $2p_k^{-1}\widetilde{U}_{12}^{k\top} \bullet (\cdots)$ all converge to 0, and it remains to discuss convergence of the fourth term $2p_k^{-1}\widetilde{U}_{22}^k \bullet (\cdots)$.

This term splits into two terms: $2p_k^{-1}\widetilde{U}_{22}^k \bullet (\cdots) = \tau_k + \sigma_k$. The second of those is

$$\begin{aligned}\sigma_k &= 2p_k^{-1}\widetilde{U}_{22}^k \bullet ((I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1}) \\ &= 2p_k^{-1}(I - p_k^{-1}G_k)_{22}^{-1}\widetilde{U}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1} \bullet \widetilde{\Psi}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{22}^k,\end{aligned}$$

which is nonnegative, because $\widetilde{U}_{22}^k \to \bar{U}_{22} \succ 0$ and $(I - p_k^{-1}G_k)_{22}^{-1} \succeq 0$ for G_{22}^k close enough to $\bar{G}_{22} = 0$, a fact which follows from wedge convergence $G(x_k) \xrightarrow{w} G(\bar{x})$. Passing to a subsequence as $k \to \infty$, the term σ_k therefore converges to some value $\sigma \geq 0$. Again, $\sigma = +\infty$ seems a priori possible, but we will be able to rule this out below.

4) Finally, the first of the terms in $2p_k^{-1}\widetilde{U}_{22}^k \bullet (\cdots)$ is

$$\begin{aligned}\tau_k &= 2p_k^{-1}\widetilde{U}_{22}^k \bullet ((I - p_k^{-1}G_k)_{22}^{-1}\widetilde{\Psi}_{12}^{k\top}(I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{12}^k(I - p_k^{-1}G_k)_{22}^{-1}) \\ &= (I - p_k^{-1}G_k)_{22}^{-1}\widetilde{U}_{22}^k(I - p_k^{-1}G_k)_{22}^{-1} \bullet 2\widetilde{\Psi}_{12}^{k\top}p_k^{-1}(I - p_k^{-1}G_k)_{11}^{-1}\widetilde{\Psi}_{12}^k.\end{aligned}$$

Now observe that $(I - p_k^{-1}G_k)_{22}^{-1} \to I_{22}$ under wedge convergence $(x_k, p_k) \xrightarrow{w} (\bar{x}, 0)$, so that the term on the left of $\bullet$ in τ_k converges to $\bar{U}_{22}$. As for the term on the right of $\bullet$, observe that

$$p_k^{-1}(I - p_k^{-1}G_k)_{11}^{-1} = (p_kI - G_k)_{11}^{-1} \longrightarrow \operatorname{diag}\left(-\frac{1}{\bar{g}_1}, \ldots, -\frac{1}{\bar{g}_s}\right),$$

which means that the term to the right of $\bullet$ converges to $2\bar{\Psi}_{12}^\top(-G(\bar{x})^\dagger)_{11}\bar{\Psi}_{12}$. Consequently, $\tau_k \to 2\bar{\Psi}_{12}^\top(-G(\bar{x})^\dagger)_{11}\bar{\Psi}_{12} \bullet \bar{U}_{22} = d^\top\mathcal{H}(\bar{x}, \bar{U})d$, using (12). We have shown that $2p_k^{-1}\Xi_k = \sigma_k + \tau_k \to \sigma + d^\top\mathcal{H}(\bar{x}, \bar{U})d$.

Altogether, in (23), passing to the limit in each of the terms, we have the following situation:

$$-\vartheta = d^\top L_{xx}(\bar{x}, \bar{U})d + d^\top\mathcal{H}(\bar{x}, \bar{U})d + \sigma$$

with $\vartheta \geq 0$, $\sigma \geq 0$. Since $d \neq 0$ is a critical direction by part 3), the second-order sufficient no-gap optimality condition implies $d^\top(L''(\bar{x}, \bar{U}) + \mathcal{H}(\bar{x}, \bar{U}))d > 0$. This is clearly a contradiction and therefore proves the result. ■

Remark. The result could be summarized by saying that $F_{xx}(x, U, p)^{-1}$ remains bounded as $p \to 0$, $x \to \bar{x}$, $U \to \bar{U}$ as long as convergence takes place in a controlled fashion, namely, as long as $(x, U, p) \xrightarrow{w} (\bar{x}, \bar{U}, 0)$. This is what originally motivated the definition of wedge convergence.

Recall the definition $Z_p(x) = (I - p^{-1}G(x))^{-1}$. We have the following

LEMMA 5. *Suppose* $(x, p) \xrightarrow{w} (\bar{x}, 0)$, *then* $Z_p(x) \to \bar{Z}$, *where* $\bar{Z} = \operatorname{diag}(0_s, I_{m-s})$.

Proof. We prove that $(x, p) \xrightarrow{w} (\bar{x}, 0)$ implies $G(x) \xrightarrow{w} G(\bar{x})$, which in turn gives $g_i(x) \xrightarrow{w} \bar{g}_i$. In other words, $(g_i(x) - \bar{g}_i)/p \to 0$. This is a consequence of the fact that eigenvalue functions of symmetric matrices are locally Lipschitz: $|\lambda_i(X) - \lambda_i(\bar{X})| \leq K\|X - \bar{X}\|$. Since the operator G is locally Lipschitz, we deduce $|\lambda_i(G(x)) - \lambda_i(G(\bar{x}))| \leq K'\|x - \bar{x}\|$, which is $|g_i(x) - \bar{g}_i| \leq K'\|x - \bar{x}\|$, proving wedge convergence $g_i(x) \xrightarrow{w} \bar{g}_i$, that is, $(g_i(x) - \bar{g}_i)/p \to 0$ as $(x, p) \xrightarrow{w} (\bar{x}, 0)$.

Now the eigenvalues of $Z_p(x) = (I - p^{-1}G(x))^{-1}$ are $z_i(x) = p/(p - g_i(x))$. This clearly gives $z_i(x) \to \bar{z}_i$ under wedge convergence, where $\bar{z}_i = 1$ for $i = 1, \ldots, s$ and $\bar{z}_i = 0$ for $i = s + 1, \ldots, m$. Using the above argument backwards, convergence of the eigenvalues

$z_i(x) \to \bar{z}_i$ under wedge convergence $(x, p) \overset{w}{\to} (\bar{x}, 0)$ implies convergence of the matrices $Z_p(x) \to \bar{Z}$. Here we use $\|X\| = \|QXQ^\top\|$ for orthogonal Q. ■

LEMMA 6. *Assuming* (H_1) –(H_3)*, there exists* $\epsilon_2 > 0$ *and* $K_2 > 0$ *such that*

$$\|F_{xU}(x, U, p)\| \leq K_2, \tag{24}$$

for every $(x, U, p) \in \mathcal{W}(\epsilon_2)$.

Proof. It suffices to write

$$F_{xU}(x, U, p)\delta U = G'(x)^\star[(I - p^{-1}G(x))^{-1}\delta U(I - p^{-1}G(x))^{-1}]$$

for a test vector $\delta U \in \mathbb{S}^m$. This shows of course that F_{xU} does not depend on U. Then convergence $Z_p(x) = (I - p^{-1}G(x))^{-1} \to \bar{Z}$ as $(x, p) \overset{w}{\to} (\bar{x}, 0)$ proved in Lemma 5 above readily implies boundedness of F_{xU} as $(x, U, p) \overset{w}{\to} (\bar{x}, \bar{U}, 0)$. ■

Let us collect some more facts about wedge convergence. We need a refinement of Lemma 5. Let $x^+ \in \mathbb{R}^n$ and write $G(x^+) = QG^+Q^\top$ for $G^+ = \text{diag}(g_1(x^+), \dots, g_m(x^+))$. Then Q also diagonalizes $Z_p(x^+)$, that is, $Z_p(x^+) = QD_p(x^+)Q^\top$, where $D_p(x^+) = \text{diag}(z_1(x^+), \dots, z_m(x^+))$. We have the following

LEMMA 7. *Decomposing Q according to* (9)*, there exists* $\epsilon_3 > 0$ *and a constant* $K_3 > 0$ *such that for all* $(x^+, p) \in \mathcal{W}'(\epsilon_3)$,

1. *The blocks of Q satisfy the following estimates:*

$$\begin{aligned} \|Q_{22}^\top Q_{22} - I\| \leq K_3\|x^+ - \bar{x}\|, \quad \|Q_{11}^\top Q_{11} - I\| \leq K_3\|x^+ - \bar{x}\|, \\ \|Q_{12}Q_{12}^\top\| \leq K_3\|x^+ - \bar{x}\|, \quad \|Q_{22}Q_{12}^\top\| \leq K_3\|x^+ - \bar{x}\|, \\ \|Q_{21}Q_{21}^\top\| \leq K_3\|x^+ - \bar{x}\|, \quad \|Q_{11}Q_{21}^\top\| \leq K_3\|x^+ - \bar{x}\|. \end{aligned} \tag{25}$$

2. *The blocks of $Z_p(x^+)$ satisfy the following estimates:*

$$\begin{aligned} \|Z_p(x^+)_{11}\| \leq K_3 p, \quad \|Z_p(x^+)_{12}\| \leq K_3\|x^+ - \bar{x}\|, \\ \|Z_p(x^+)_{22} - I\| \leq K_3 p^{-1}\|x^+ - \bar{x}\|. \end{aligned} \tag{26}$$

Proof. Let us start by writing

$$G(x^+) - G(\bar{x}) = Q(G^+ - \bar{G})Q^\top + Q\bar{G}Q^\top - \bar{G},$$

where Q diagonalizes $G(x^+)$ with diagonal matrix G^+. We can see that the first term on the right-hand side is $\mathcal{O}(G(x^+) - G(\bar{x}))$, because eigenvalue functions are locally Lipschitz, and because $\|X\| = \|QXQ^\top\|$. Subtracting this term, shows $Q\bar{G}Q^\top - \bar{G} = \mathcal{O}(G(x^+) - G(\bar{x}))$. But G is locally Lipschitz, so we have $Q\bar{G}Q^\top - \bar{G} = \mathcal{O}(\|x^+ - \bar{x}\|)$. Expanding this term gives

$$Q\bar{G}Q^\top - \bar{G} = \begin{bmatrix} Q_{11}\bar{G}_{11}Q_{11}^\top - \bar{G}_{11} & Q_{11}\bar{G}_{11}Q_{21}^\top \\ Q_{21}\bar{G}_{11}Q_{11}^\top & Q_{21}\bar{G}_{11}Q_{21}^\top \end{bmatrix}$$

using (7). This implies three estimates, namely $Q_{21}\bar{G}_{11}Q_{21}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$ for the $(2, 2)$ block, $Q_{21}\bar{G}_{11}Q_{11}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$ for the off-diagonal blocks, and $Q_{11}\bar{G}_{11}Q_{11}^\top - \bar{G}_{11} = \mathcal{O}(\|x^+ - \bar{x}\|)$ for the $(1, 1)$ block.

Since $\bar{G}_{11} \prec 0$, $Q_{21}\bar{G}_{11}Q_{21}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$ implies $Q_{21}Q_{21}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$. For the same reason, $Q_{11}\bar{G}_{11}Q_{11}^\top - \bar{G}_{11} = \mathcal{O}(\|x^+ - \bar{x}\|)$ implies $Q_{11}Q_{11}^\top - I = \mathcal{O}(\|x^+ - \bar{x}\|)$ and similarly, $Q_{21}\bar{G}_{11}Q_{11}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$ implies $Q_{21}Q_{11}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$.

From orthogonality of Q we can deduce three more things. Namely, $Q_{11}Q_{11}^\top + Q_{12}Q_{12}^\top = I$ implies $Q_{12}Q_{12}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$, $Q_{21}Q_{11}^\top + Q_{22}Q_{12}^\top = 0$ implies $Q_{22}Q_{12}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$, and thirdly $Q_{21}Q_{21}^\top + Q_{22}Q_{22}^\top = I$ implies $Q_{22}Q_{22}^\top - I = \mathcal{O}(\|x^+ - \bar{x}\|)$. That completes the list of statements in (25).

Consider item 2. Recall that $Z_p(x^+) \to \bar{Z}$ under wedge convergence $(x^+, p) \stackrel{w}{\to} (\bar{x}, 0)$ by Lemma 5. Looking at the diagonal matrix $D_p(x^+)$ associated with $Z_p(x^+)$, we can see that $D_p(x^+)_{11} \to 0$ with speed $D_p(x^+)_{11} = \mathcal{O}(p)$. This follows from the estimate

$$|z_i(x^+)| = \frac{p}{p - g_i(x^+)} \le \frac{p}{-g_i(x^+)} \le Kp, \quad i = 1, \ldots, s,$$

which uses $g_i(x^+) \to \bar{g}_i < 0$ for $i = 1, \ldots, s$.

Similarly, we have $D_p(x^+)_{22} \to I$ under wedge convergence $(x^+, p) \stackrel{w}{\to} (\bar{x}, 0)$ with speed $D_p(x^+)_{22} - I = \mathcal{O}(p^{-1}\|x^+ - \bar{x}\|)$. Here we use the estimate

$$|z_i(x^+) - 1| \le \frac{|g_i(x^+)|}{p - g_i(x^+)} = \frac{|g_i(x^+) - \bar{g}_i|/p}{1 - (g_i(x^+) - \bar{g}_i)/p}, \quad i = s+1, \ldots, m,$$

where $\bar{g}_i = 0$ for $i = s+1, \ldots, m$. Then the denominator is bounded on a wedge neighborhood, while the nominator is of the order $\mathcal{O}(p^{-1}\|x^+ - \bar{x}\|)$.

Using these estimates, observe now that we have $Z_p(x^+)_{11} = Q_{11}D_{11}Q_{11}^\top + Q_{12}D_{22}Q_{12}^\top$. Since $Q_{12}Q_{12}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$ and $D_{22} \to I$ under wedge convergence $(x^+, p) \stackrel{w}{\to} (\bar{x}, 0)$, the second term on the right is $\mathcal{O}(\|x^+ - \bar{x}\|)$. Since $Q_{11}Q_{11}^\top - I = \mathcal{O}(\|x^+ - \bar{x}\|)$, the first term on the right is $\mathcal{O}(\|D_{11}\|) = \mathcal{O}(p)$ under wedge convergence. That proves the estimate $Z_p(x^+)_{11} = \mathcal{O}(p + \|x^+ - \bar{x}\|)$. Under wedge convergence we have $\|x^+ - \bar{x}\| = p(\|x^+ - \bar{x}\|/p) = o(p)$, so the dominant expression is $Z_p(x^+)_{11} = \mathcal{O}(p)$ as claimed. This proves the first estimate in item 3.

Next observe that we have $Z_p(x^+)_{12} = Q_{11}D_{11}Q_{21}^\top + Q_{12}D_{22}Q_{22}^\top$. The first term is $\mathcal{O}(p\|x^+ - \bar{x}\|)$, because as we have seen $D_{11} = \mathcal{O}(p)$, and $Q_{11}Q_{21}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$ according to item 1. The second term $Q_{12}D_{22}Q_{22}^\top$ on the other hand can be written as $Q_{12}(D_{22} - I)Q_{22}^\top + Q_{12}Q_{22}^\top$. Here $Q_{12}(D_{22} - I)Q_{22}^\top = \mathcal{O}(D_{22} - I)\mathcal{O}(Q_{12}Q_{22}^\top) = \mathcal{O}(p^{-1}\|x^+ - \bar{x}\|^2)$. On the other hand, $Q_{12}Q_{22}^\top = I - Q_{11}Q_{11}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$ by orthogonality of Q and by item 1. Since $\|x^+ - \bar{x}\|/p \to 0$ by the definition of wedge convergence, the dominant term for $Z_p(x^+)_{12}$ is of the order $\mathcal{O}(\|x^+ - \bar{x}\|)$ as claimed.

Finally, we have $Z_p(x^+)_{22} - I = Q_{21}D_{11}Q_{21}^\top + Q_{22}D_{22}Q_{22}^\top - I$. Since $Q_{11}Q_{11}^\top - I = \mathcal{O}(\|x^+ - \bar{x}\|)$, orthogonality gives $Q_{21}Q_{21}^\top = \mathcal{O}(\|x^+ - \bar{x}\|)$, so that the first term is $\mathcal{O}(p\|x^+ - \bar{x}\|)$. Let us write the second term as $Q_{22}(D_{22} - I)Q_{22}^\top + Q_{22}Q_{22}^\top - I$. Here the left-hand term is $\mathcal{O}(p^{-1}\|x^+ - \bar{x}\|)$, while the right-hand term is $\mathcal{O}(\|x^+ - \bar{x}\|)$. The dominant expression is the first one, proving the estimate. ■

Recall that $U^+(x^+, U, p) = Z_p(x^+)UZ_p(x^+)$. We have the following

LEMMA 8. *There exists a wedge neighborhood $\mathcal{W}(\epsilon_4)$ and a constant $K_4 > 0$ such that*

$$\|U_{11}^+(x^+, U, p)\| \le K_4(p^2\|U - \bar{U}\| + \|x^+ - \bar{x}\|^2), \tag{27}$$

and

$$\|U_{12}^+(x^+, U, p)\| \le K_4(p\|U - \bar{U}\| + \|x^+ - \bar{x}\|). \tag{28}$$

for every $(x^+, U, p) \in \mathcal{W}(\epsilon_4)$.

Proof. Notice first that

$$U_{11}^{+} = Z_p(x)_{11}(U_{11}Z_p(x)_{11} + U_{12}Z_p(x)_{12}^{\top}) + Z_p(x)_{12}(U_{12}^{\top}Z_p(x)_{11} + U_{22}Z_p(x)_{12}^{\top}).$$

There are four terms to discuss here. Using (26) in the previous lemma, and observing $\bar{U}_{11} = 0$, we have $U_{11} = \mathcal{O}(\|U - \bar{U}\|)$, so the first term $Z_{11}U_{11}Z_{11}$ is $\mathcal{O}(p^2\|U - \bar{U}\|)$. Using $\bar{U}_{12} = 0$ and (26), the second term $Z_{11}U_{12}Z_{12}^{\top}$ is $\mathcal{O}(p\|U - \bar{U}\|\|x^+ - \bar{x}\|)$. Since $\|x^+ - \bar{x}\|/p \to 0$ under wedge convergence, the second term is therefore even $o(p^2\|U - \bar{U}\|)$. By symmetry, the same applies to the third term. As for the fourth term $Z_{12}U_{22}Z_{12}^{\top}$, notice that $U_{22} \to \bar{U}_{22} \succ 0$, so $U_{22} = \mathcal{O}(1)$. From (26) we therefore obtain an expression of the form $\mathcal{O}(\|x^+ - \bar{x}\|^2)$. That proves the first estimate (27).

Similarly, the second estimate comprises four terms:

$$U_{12}^{+\top} = Z_p(x)_{12}^{\top}(U_{11}Z_p(x)_{11} + U_{12}Z_p(x)_{12}^{\top}) + Z_p(x)_{22}(U_{12}^{\top}Z_p(x)_{11} + U_{22}Z_p(x)_{12}^{\top}).$$

Using again $\bar{U}_{11} = 0$, $\bar{U}_{12} = 0$, so that $U_{11} = \mathcal{O}(\|U - \bar{U}\|)$ and $U_{12} = \mathcal{O}(\|U - \bar{U}\|)$, while $U_{22} = \mathcal{O}(1)$, the previous Lemma 7 gives $\mathcal{O}(p\|x^+ - \bar{x}\|\|U - \bar{U}\|)$ for the first term, which is $o(p^2\|U - \bar{U}\|)$ under wedge convergence. The second term $Z_{12}^{\top}U_{12}Z_{12}^{\top}$ is $\mathcal{O}(\|x^+ - \bar{x}\|^2\|U - \bar{U}\|) = o(p^2\|U - \bar{U}\|)$ under wedge convergence. The third term $Z_{22}U_{12}^{\top}Z_{11}$ is $\mathcal{O}(p\|U - \bar{U}\|)$, because $Z_{22} = \mathcal{O}(1)$. The last term $Z_{22}U_{22}Z_{12}^{\top}$ is $\mathcal{O}(\|x^+ - \bar{x}\|)$, because $U_{22} = \mathcal{O}(1)$. This gives the two dominant terms in (28). ■

9. Lemmas with wedge convergence II

In this section we consider two more technical results, which use the concept of wedge convergence, in tandem with the no-gap second-order optimality condition.

LEMMA 9. *Assume hypotheses* (H_1)–(H_3)*, let* $U^+(x^+, U, p) = Z_p(x^+)UZ_p(x^+)$*, and write* U^+ *for short. There exists a wedge neighborhood* $\mathcal{W}(\epsilon_5)$*, a neighborhood* N *of* $\bar{U}$*, and a constant* $K_5 > 0$ *such that the following condition is satisfied. Suppose* $(x^+, U, p) \in \mathcal{W}(\epsilon_5)$*,* $U^+ \in N$*, and* $\delta U \in \mathbb{S}^m$ *with* $\|\delta U\| = 1$*, and put* $h = p^{-1}F_{xx}^{-1}(x^+, U, p)F_{xU}(x^+, U, p)\delta U$*. Suppose the exotic equation*

$$\begin{aligned} h^{\top}L_{xx}(x^+, U^+)h &+ p^{-1}[G'(x^+)h] \bullet ([I - p^{-1}G(x^+)]^{-1}\delta U[I - p^{-1}G(x^+)]^{-1}) \\ &+ 2p^{-1}[G'(x^+)h] \bullet ([I - p^{-1}G(x^+)]^{-1}[G'(x^+)h] \\ &\times [I - p^{-1}G(x^+)]^{-1}U[I - p^{-1}G(x^+)]^{-1}) = 0 \end{aligned} \tag{29}$$

is satisfied. Then $\|h\| \leq K_5$.

Proof. Suppose contrary to the statement that there exist $(x_k^+, U_k, p_k) \xrightarrow{w} (\bar{x}, \bar{U}, 0)$ and $U_k^+ \to \bar{U}$ along with unit vectors δU_k such that equation (29) is satisfied, but$\|h_k\| \to \infty$, where

$$h_k = p_k^{-1}F_{xx}^{-1}(x_k^+, U_k, p_k)F_{xU}(x_k^+, U_k, p_k)\delta U_k.$$

Put $d_k = h_k/\|h_k\|$. Passing to a subsequence, we may assume that $d_k \to d$ for a unit vector d, and also $\delta U_k \to \delta U$ for a unit vector δU.

Dividing (29) by $\|h_k\|^2$ gives

$$d_k^\top L_{xx}(x_k^+, U_k^+)d_k + [G'(x_k^+)d_k] \bullet \left([I - p_k^{-1}G(x_k^+)]^{-1}\frac{\delta U_k}{p_k\|h_k\|}[I - p_k^{-1}G(x_k^+)]^{-1}\right)$$
$$+ 2p_k^{-1}[G'(x_k^+)d_k] \bullet ([I - p_k^{-1}G(x_k^+)]^{-1}[G'(x_k^+)d_k]$$
$$\times [I - p_k^{-1}G(x_k^+)]^{-1}U_k[I - p_k^{-1}G(x_k^+)]^{-1}) = 0. \tag{30}$$

There are now two cases to be discussed. Case 1 is when $p_k\|h_k\| \geq \mu > 0$ for some μ and a subsequence of $k \in \mathcal{K}$. Case 2 is when $p_k\|h_k\| \to 0$.

Let us discuss case 1 first. Considering the subsequence $k \in \mathcal{K}$ only, the term

$$\Theta_k = [G'(x_k^+)d_k] \bullet \left([I - p_k^{-1}G(x_k^+)]^{-1}\frac{\delta U_k}{p_k\|h_k\|}[I - p_k^{-1}G(x_k^+)]^{-1}\right)$$

is bounded above by $\mu^{-1}[G'(x_k^+)d_k] \bullet ([I - p_k^{-1}G(x_k^+)]^{-1}\delta U_k[I - p_k^{-1}G(x_k^+)]^{-1})$, which is bounded on a wedge neighborhood. Passing to yet another subsequence, and using Lemma 5, we may therefore assume $\Theta_k \to \Theta$ for some $\Theta \in \mathbb{R}$. Going back with this information to (30), we see that the identity is now of the form

$$d_k^\top L_{xx}(x_k^+, U_k^+)d_k + \Theta_k + 2p_k^{-1}\Xi_k = 0,$$

where the two leftmost terms converge. Consequently, $2p_k^{-1}\Xi_k$ has no choice, it converges, and given the fact that $p_k^{-1} \to \infty$, this implies $\Xi_k \to 0$. Now

$$\Xi_k = [G'(x_k^+)d_k] \bullet (Z_{p_k}(x_k^+)[G'(x_k^+)d_k]Z_{p_k}(x_k^+)U_kZ_{p_k}(x_k^+))$$

converges to

$$\Xi = [G'(\bar{x})d]_{22} \bullet [G'(\bar{x})d]_{22}\bar{U}_{22} = 0,$$

where we use $Z_{p_k}(x_k^+) \to \bar{Z} = \mathrm{diag}(0_s, I_{m-s})$ by Lemma 5. Since $\bar{U}_{22} \succ 0$ by strict complementarity, we deduce $[G'(\bar{x})d]_{22} = 0$, that is, d is a critical direction (10).

Let us analyze the term Θ_k in (30) a little further. Writing $Z_k := Z_{p_k}(x_k^+)$ and using (20) in tandem with the definition $d_k = p_k^{-1}\|h_k\|^{-1}F_{xx}^{-1}F_{xU}\delta U_k$ gives

$$\Theta_k = p_k^{-2}\|h_k\|^{-2}[G'(x_k^+)F_{xx}^{-1}G'(x_k^+)^\star(Z_k\delta U_kZ_k)] \bullet (Z_k\delta U_kZ_k).$$

Since the quadratic form $G'(x_k^+)F_{xx}^{-1}G'(x_k^+)^\star$ is positive semidefinite by Lemma 4, this implies $\Theta_k \geq 0$. Passing to a subsequence, we may therefore assume that $\Theta_k \to \Theta$, where $\Theta \geq 0$.

As we know from the proof of Lemma 4, the term $2p_k^{-1}\Xi_k = 2p_k^{-1}Z_k\Psi^kZ_k\Psi^kZ_k \bullet U_k$ in (30) may be decomposed as $\sigma_k + \tau_k$, where $\sigma_k \geq 0$, and $\tau_k \to d^\top\mathcal{H}(\bar{x}, \bar{U})d$. We therefore have the following situation:

$$d_k^\top L_{xx}(x_k^+, U_k^+)d_k + \Theta_k + \sigma_k + \tau_k = 0.$$

Passing to the limit, we find that

$$d^\top L_{xx}(\bar{x}, \bar{U})d + \Theta + \sigma + d^\top\mathcal{H}(\bar{x}, \bar{U})d = 0.$$

Since $\Theta + \sigma \geq 0$ and d is a critical direction, this contradicts the second-order sufficient no-gap optimality condition (hypothesis (H_2)) and settles case 1.

Let us now consider case 2, where $\|h_k\| \to \infty$, but $p_k\|h_k\| \to 0$. Multiplying (30) with $p_k\|h_k\|$ gives the identity

$$p_k\|h_k\|d_k^\top L_{xx}(x_k^+, U_k^+)d_k + [G'(x_k^+)d_k] \bullet ([I - p_k^{-1}G(x_k^+)]^{-1}\delta U_k[I - p_k^{-1}G(x_k^+)]^{-1}) + 2\|h_k\|[G'(x_k^+)d_k] \bullet ([I - p_k^{-1}G(x_k^+)]^{-1}[G'(x_k^+)d_k] \times [I - p_k^{-1}G(x_k^+)]^{-1}U_k[I - p_k^{-1}G(x_k^+)]^{-1}) = 0. \tag{31}$$

Here the first term converges to 0, the second term $\widetilde{\Theta}_k = [G'(x_k^+)d_k] \bullet (\cdots)$ converges to $\widetilde{\Theta} := [G'(\bar{x})d]_{22} \bullet \delta U_{22}$. Therefore, the rightmost term in (31) is also convergent.

This term is now of the form $2\|h_k\|\Xi_k$, where Ξ_k is as before, and $\|h_k\| \to \infty$. Therefore, we must have $\Xi_k \to 0$. But $\Xi_k \to \Xi = \bar{\Psi} \bullet \bar{Z}\bar{\Psi}\bar{Z}\bar{U}\bar{Z} = \bar{U}_{22} \bullet \bar{\Psi}_{22}\bar{\Psi}_{22} = 0$. Since $\bar{U}_{22} \succ 0$ by strict complementarity, this implies $\bar{\Psi}_{22} = [G'(\bar{x})d]_{22} = 0$, so that d is a critical direction (10).

Using this information, we now go back to (30). Here the third term is of the form $2p_k^{-1}\Xi_k$. Using the argument in the proof of Lemma 4, we have $2p_k^{-1}\Xi_k = \sigma_k + \tau_k$, where $\sigma_k \geq 0$ and $\tau_k \to d^\top\mathcal{H}(\bar{x}, \bar{U})d$. Let us examine the second term of (30), which is $\Theta_k = p_k^{-1}\|h_k\|^{-1}\Psi^k \bullet Z_k\delta U_k Z_k$. Substituting backwards, using $d_k = h_k/\|h_k\|$, the definition of h_k, and representing F_{xU} as in (20), we have

$$\begin{aligned}\Theta_k &= p_k^{-2}\|h_k\|^{-2}[G'(x_k^+)h] \bullet Z_k\delta U_k Z_k \\ &= p_k^{-2}\|h_k\|^{-2}[G'(x_k^+)F_{xx}(x_k^+, U_k, p_k)^{-1}G'(x_k^+)^\star(Z_k\delta U_k Z_k)] \bullet (Z_k\delta U_k Z_k) \geq 0,\end{aligned}$$

the latter, because the quadratic form $G'(x_k^+)F_{xx}^{-1}G'(x_k^+)^\star$ is positive semidefinite by Lemma 4. This means $\Theta_k \geq 0$. We therefore find the following situation:

$$d_k^\top L_{xx}(x_k^+, U_k^+)d_k + \Theta_k + \tau_k + \sigma_k = 0,$$

which after passing to a subsequence converges to the limit $d^\top L_{xx}(\bar{x}, \bar{U})d + \Theta + d^\top\mathcal{H}(\bar{x}, \bar{U})d + \sigma = 0$. This contradicts the second-order optimality condition, because $\Theta + \sigma \geq 0$, and since d was recognized as a critical direction. This ends case 2, and thereby completes the proof. ■

Recall the notation $U^+ = U^+(x^+, U, p) = Z_p(x^+)UZ_p(x^+)$ in (19), where $Z_p(x^+) = (I - p^{-1}G(x^+))^{-1}$. We have the following technical

LEMMA 10. *Under hypotheses* (H_1)–(H_3), *there exist* $\epsilon_6 > 0$ *and a constant* $K_6 > 0$, *such that the following condition is satisfied*: *Suppose* $(x^+, U, p) \in \mathcal{W}(\epsilon_6)$ *and* $\delta U \in \mathbb{S}^m$ *with* $\|\delta U\| = 1$ *are such that*

$$h := p^{-1}F_{xx}(x^+, U, p)^{-1}F_{xU}(x^+, U, p)\delta U,$$

and

$$H := p^{-1}Z_p(x^+)\delta U Z_p(x^+) + p^{-1}U^+[G'(x^+)h]Z_p(x^+) + *$$

satisfy the equation

$$L_{xx}(x^+, U^+)h + G'(x^+)^\star H = 0. \tag{32}$$

Then $\|H_{22}\| \leq K_6(\|h\| + 1)$.

Proof. Let us write (32) as

$$L_{xx}(x^+, U^+)h + (G_1(x^+) \bullet H, \dots, G_n(x^+) \bullet H) = 0.$$

Using the decomposition (9), and shifting (1, 1) and (1, 2)-terms to the right, this becomes

$$G_j(x^+)_{22} \bullet H_{22} = -e_j^\top L_{xx}(x^+, U^+)h - G_j(x^+)_{11} \bullet H_{11} - 2\mathrm{Tr}(G_j(x^+)_{12} H_{12}^\top)$$

where e_j is the jth coordinate unit vector. Therefore each $G_j(x^+)_{22} \bullet H_{22}$ is of the form $\mathcal{O}(\|h\| + \|H_{12}\| + \|H_{11}\|)$, because U^+ is bounded on a wedge neighborhood by Lemma 8. Now by the LICQ hypothesis (H_3), the operator (15) is injective, and therefore $W \mapsto (G_1(x^+)_{22} \bullet W, \dots, G_n(x^+)_{22} \bullet W)$ is injective at x^+ in a neighborhood of $\bar{x}$. In other words, $\|(G_1(x^+)_{22} \bullet W, \dots, G_n(x^+)_{22} \bullet W)\| \geq \kappa \|W\|$ for some $\kappa > 0$, all W, and all x^+ sufficiently close to $\bar{x}$. This proves $H_{22} = \mathcal{O}(\|h\| + \|H_{12}\| + \|H_{11}\|)$.

Next observe that by the definition of H,

$$\begin{aligned} H_{11} &= (p^{-1} Z_p(x^+) \delta U Z_p(x^+))_{11} + (p^{-1} U^+ [G'(x^+)h] Z_p(x^+))_{11} + * \\ &= p^{-1} Z_p(x^+)_{11} \delta U_{11} Z_p(x^+)_{11} + p^{-1}(U_{11}^+ \Psi_{11} Z_p(x^+)_{11} + U_{12}^+ \Psi_{12}^\top Z_p(x^+)_{11}) + *, \end{aligned}$$

where we have put $\Psi = G'(x^+)h$ for brevity. According to (27) we have $U_{11}^+ = \mathcal{O}(1)$, while $p^{-1} Z_p(x^+)_{11} = \mathcal{O}(1)$ under wedge convergence by (26). Similarly $U_{12}^+ = \mathcal{O}(1)$ by (28). Putting these together therefore gives $H_{11} = \mathcal{O}(1 + \|h\|)$ under wedge convergence. The same applies to H_{12}:

$$H_{12} = p^{-1} Z_p(x^+)_{11} \delta U_{12} Z_p(x^+)_{22} + p^{-1}(Z_p(x^+)_{11} \Psi_{12} U_{22}^+ + Z_p(x^+)_{11} \Psi_{11} U_{12}^+) + *.$$

This completes the proof. ■

10. Application of the implicit function theorem

Let us now put $\epsilon_7 = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6\}$. Then all the properties collected over the previous Lemmas will be valid on the wedge neighborhood $\mathcal{W}(\epsilon_7)$.

Next consider the system of nonlinear equations

$$F_x(x, U, p) = 0,$$

based on formula (16). Notice that $(\bar{x}, \bar{U}, p)$ is solution for every $p > 0$. Let us fix an interval $\mathcal{I} = [p_1, p_2]$ such that $0 < p_1 < p_2 \leq \epsilon_7$. We apply the implicit function theorem Lemma 1 where the H in the Lemma becomes F_x, the compact set is $K^* = \{\bar{U}\} \times \mathcal{I}$, the variable y is $(U, p) \in \mathbb{S}^m \times \mathbb{R}$, while x is x. The invertibility hypothesis on H_x in Lemma 1 therefore reduces to invertibility of F_{xx}, which is guaranteed by Lemma 4 (22). Consequently, there exists an open neighborhood $\mathcal{M}_{p_1,p_2} \subset \mathbb{R}^n \times \mathbb{S}^m \times \mathbb{R}$ of $\{\bar{x}\} \times \{\bar{U}\} \times [p_1, p_2]$, an open neighborhood $\mathcal{N}_{p_1,p_2}$ of $\{\bar{U}\} \times [p_1, p_2]$ in $\mathbb{S}^m \times \mathbb{R}$, and a C^1 function $x^+(\cdot,\cdot)\colon \mathcal{N}_{p_1,p_2} \to \mathbb{R}^n$ such that $F_x(x^+(U, p), U, p) = 0$ for every $(U, p) \in \mathcal{N}_{p_1,p_2}$, $x^+(\bar{U}, p) = \bar{x}$ for all $p \in [p_1, p_2]$, and such that the function $x^+(\cdot,\cdot)$ is unique in the sense that $(x, U, p) \in \mathcal{M}_{p_1,p_2}$ together with

$F_x(x, U, p) = 0$ implies $x = x^+(U, p)$. This may also be expressed by

$$\left\{(x, U, p) \in \mathcal{M}_{p_1,p_2} : F_x(x, U, p) = 0\right\} = \left\{\left(x^+(U, p), U, p\right) : (U, p) \in \mathcal{N}_{p_1,p_2}\right\}. \quad (33)$$

We may assume without loss that

$$\mathcal{M}_{p_1,p_2} \subset \mathcal{W}(\epsilon_7) \text{ for every } \mathcal{I} = [p_1, p_2] \text{ having } p_2 \le \epsilon_7, \quad (34)$$

because $\mathcal{W}(\epsilon_7)$ is a neighborhood of $\{\bar{x}\} \times \{\bar{U}\} \times [p_1, p_2]$. Moreover, by further shrinking $\mathcal{M}_{p_1,p_2}$ and $\mathcal{N}_{p_1,p_2}$ if required, we may arrange that

$$\begin{cases} \mathcal{M}_{p_1,p_2} \text{ is of the form } \mathcal{M}_{p_1,p_2} = \mathcal{U}_{p_1,p_2} \times \mathcal{N}_{p_1,p_2} \\ \text{for an open neighborhood } \mathcal{U}_{p_1,p_2} \text{ of } \bar{x}. \end{cases} \quad (35)$$

and similarly that

$$\begin{cases} \mathcal{N}_{p_1,p_2} \text{ is of the form } \mathcal{N}_{p_1,p_2} = N_{p_1,p_2} \times I_{p_1,p_2} \text{ for a convex open} \\ \text{neighborhood } N_{p_1,p_2} \text{ of } \bar{U} \text{ and an open interval } I_{p_1,p_2} \text{ containing} \\ \mathcal{I} = [p_1, p_2]. \end{cases} \quad (36)$$

In our notation $x^+(U, p)$ we have suppressed the dependence of the implicit function on the choice of $\mathcal{I} = [p_1, p_2]$, but we will say that $x^+(U, p)$ is associated with the choice of some $\mathcal{I}$. This slight abuse of notation is justified by the following

LEMMA 11. *Under the standing assumptions* (H_1)–(H_3), *we have the following uniqueness statements*:

1. *Suppose* $0 < p_1 < p_2 \le \epsilon_7$ *and* $(U, p) \in \mathcal{N}_{p_1,p_2}$. *Then* $x^+(U, p)$ *is the unique local minimum (even the unique critical point) of program* $\min_{x\in\mathbb{R}^n} F(x, U, p)$ *in the neighborhood* $\mathcal{U}_{p_1,p_2}$ *of* $\bar{x}$.
2. *Suppose* $0 < p' < p_2$ *and* $0 < p'' < p_2$, $p_2 \le \epsilon_7$ *and that we have* $(U, p) \in \mathcal{N}_{p',p_2} \cap \mathcal{N}_{p'',p_2}$. *Then the values* $x^+(U, p)$ *of the two implicit functions associated with* $[p', p_2]$ *and* $[p'', p_2]$ *agree.*

Proof. Let us prove statement (1). We first show that $x^+(U, p)$ is a local minimum of F. Clearly it is a critical point by the implicit function theorem, but in addition, we have $F_{xx}(x^+(U, p), U, p) \succeq \rho I \succ 0$, because $(x^+(U, p), U, p) \in \mathcal{W}(\epsilon_7)$ by construction, so Lemma 4 (22) applies. Now the sufficient second order optimality condition for program (5) is satisfied at $x^+(U, p)$, which is therefore a local minimum.

Suppose now x is a critical point of program $\min_{x\in\mathbb{R}^n} F(x, U, p)$ in $\mathcal{U}_{p_1,p_2}$. Then $(x, U, p) \in \mathcal{U}_{p_1,p_2} \times \mathcal{N}_{p_1,p_2} = \mathcal{M}_{p_1,p_2}$, and of course $F_x(x, U, p) = 0$. Due to formula (33), this implies $x = x^+(U, p)$.

The proof of statement (2) is based on the same argument. ■

We will make use of the derivative formula for the implicit function, which is part of the statement of the implicit function theorem (Lemma 1). Using (20), we have

$$\begin{aligned} x_U^+(U, p)\delta U &= -F_{xx}\left(x^+(U, p), U, p\right)^{-1} F_{xU}\left(x^+(U, p), U, p\right)\delta U \\ &= -F_{xx}\left(x^+(U, p), U, p\right)^{-1} G'(x)^\star \Big[\left(I - p^{-1}G\left(x^+(U, p)\right)\right)^{-1} \\ &\qquad \delta U \left(I - p^{-1}G\left(x^+(U, p)\right)\right)^{-1}\Big], \end{aligned} \quad (37)$$

whenever the implicit terms are defined.

Let us introduce a second implicit function $U^+(U, p)$ defined on $\mathcal{N}_{p_1,p_2}$ by

$$U^+(U, p) = \left(I - p^{-1}G\left(x^+(U, p)\right)\right)^{-1} U \left(I - p^{-1}G\left(x^+(U, p)\right)\right)^{-1}.$$

In other words, $U^+(U, p) = U^+\left(x^+(U, p), U, p\right)$, where the right-hand term uses the function U^+ introduced in section 8. We then have the following

Lemma 12. *Let* $0 < p_1 < p_2 \le \epsilon_7$. *Then the implicit function* $x^+(U, p)$ *associated with the interval* $\mathcal{I} = [p_1, p_2]$ *satisfies*

$$\|x_U^+(U, p)\| \le K_5 p, \tag{38}$$

for every $(U, p) \in \mathcal{N}_{p_1,p_2}$. *Similarly, the implicit function* $U^+(U, p)$ *associated with* $\mathcal{I}$ *satisfies*

$$\|\left(U_U^+(U, p)\right)_{22}\| \le K_6(K_5 + 1)p \tag{39}$$

for every $(U, p) \in \mathcal{N}_{p_1,p_2}$.

Proof. 1) We start out with formula (37). Write for brevity $x^+ = x^+(U, p)$ and put $h = p^{-1} x_U^+(U, p)\delta U = p^{-1}F_{xx}(x^+, U, p)^{-1}F_{xU}(x^+, U, p)\delta U$. The construction of the implicit function guarantees that $(x^+, U, p) \in \mathcal{W}(\epsilon_7)$ for $(U, p) \in \mathcal{N}_{p_1,p_2}$.

2) We claim that the exotic equation (29) is satisfied. This can be seen as follows. We consider the identities:

$$\begin{cases} L_x\left(x^+(U, p), U^+(U, p)\right) = 0 \\ \left(I - p^{-1}G\left(x^+(U, p)\right)\right)^{-1} U \left(I - p^{-1}G\left(x^+(U, p)\right)\right)^{-1} - U^+(U, p) = 0 \end{cases} \tag{40}$$

based on (18), $F_x = 0$, and (19). We differentiate these equations with respect to U. For the first equation in (40) we obtain

$$L_{xx}\left(x^+(U, p), U^+(U, p)\right) x_U^+(U, p)\delta U + G'\left(x(U, p)\right)^\star \left[U_U^+(U, p)\delta U\right] = 0. \tag{41}$$

Differentiating the second equation in (40) gives

$$\begin{aligned} &\left[I - p^{-1}G\left(x^+(U, p)\right)\right]^{-1} \delta U \left[I - p^{-1}G\left(x^+(U, p)\right)\right]^{-1} \\ &\quad + U^+(U, p)\left[p^{-1}G'\left(x^+(U, p)\right)\left\{x_U^+(U, p)\delta U\right\}\right]\left[I - p^{-1}G\left(x^+(U, p)\right)\right]^{-1} \\ &\quad + \left[I - p^{-1}G\left(x^+(U, p)\right)\right]^{-1}\left[p^{-1}G'\left(x^+(U, p)\right)\left\{x_U^+(U, p)\delta U\right\}\right]U^+(U, p) \\ &\quad - U_U^+(U, p)\delta U = 0. \end{aligned} \tag{42}$$

Substituting (42) into (41) gives

$$\begin{aligned} &L_{xx}\left(x^+, U^+\right)x_U\delta U + G'(x^+)^\star\left(\left[I - p^{-1}G\left(x^+\right)\right]^{-1}\delta U\left[I - p^{-1}G\left(x^+\right)\right]^{-1}\right) \\ &\quad + 2p^{-1}G'(x^+)^\star\left(\left[I - p^{-1}G\left(x^+\right)\right]^{-1}\left\{G'(x^+)x_U\delta U\right\}\left[I - p^{-1}G\left(x^+\right)\right]^{-1}\right. \\ &\quad \left.\times\, U\left[I - p^{-1}G\left(x^+\right)\right]^{-1}\right) = 0, \end{aligned} \tag{43}$$

where we write $x^+ = x^+(U, p)$, $U^+ = U^+(U, p)$ and where we suppress the arguments.

Multiplying (43) from the left with h defined in part 1) above, and dividing by p^2, we obtain indeed the exotic equation (29). In consequence, Lemma 9 applies and gives $p^{-1}\|x_U^+(U, p)\| = \|h\| \le K_5$ on $\mathcal{N}_{p_1,p_2}$.

3) Let us proceed in a similar way for the implicit function U^+. Observe that (43) is nothing else but equation (29), when we substitute the expression for h used in 1), and when we put $H = p^{-1}U_U^+(U, p)\delta U$. Therefore, Lemma 10 implies

$$\|p^{-1}\left(U_U^+(U, p)\right)_{22}\| = \|H_{22}\| \le K_6\left(\|h\| + 1\right) \le K_6(K_5 + 1).$$

This proves the second part of the statement. ■

Remark. The important fact about the constants collected over the past Lemmas is that they are independent of the choice of the interval $\mathcal{I} = [p_1, p_2]$, as long as $p_2 \le \epsilon_7$ is respected. We refer to this as prior information, because it is needed before we ultimately fix the interval $\mathcal{I}$. This will become clear in section 11.

We are now ready to obtain the following major step toward the local convergence of the AL algorithm.

LEMMA 13. *Under hypotheses* (H_1)–(H_3)*, there exists* $\epsilon_7 > 0$ *and* $K_7 > 0$ *such that for all* $0 < p_1 < p_2 \le \epsilon_7$ *the implicit functions* x^+ *and* U^+ *associated with the interval* $\mathcal{I} = [p_1, p_2]$ *satisfy the estimates*

$$(a)\ \|x^+(U, p) - \bar{x}\| \le K_7 p\|U - \bar{U}\|, \quad (b)\ \|U^+(U, p) - \bar{U}\| \le K_7 p\|U - \bar{U}\| \tag{44}$$

for every $(U, p) \in \mathcal{N}_{p_1,p_2}$.

Proof. Given the fact that $x^+(\bar{U}, p) = \bar{x}$ for every p and each of the implicit functions, we can integrate and obtain

$$\|x^+(U, p) - \bar{x}\| = \left\| \int_0^1 x_U^+(\bar{U} + \tau(U - \bar{U}), p)(U - \bar{U})\, d\tau \right\| \le K_5 p\|U - \bar{U}\|,$$

using estimate (38) in Lemma 12, $(\bar{U}, p), (U, p) \in \mathcal{N}_{p_1,p_2}$, and the fact that $\mathcal{N}_{p_1,p_2}$ is convex. This proves estimate (a) with constant K_5.

To prove estimate (b) for the multiplier update, U^+, we first apply the same argument to the $(2, 2)$-block of U^+. Since $U^+(\bar{U}, p) = \bar{U}$ for every p, we have

$$\|U^+(U, p)_{22} - \bar{U}_{22}\| \le K_6(K_5 + 1)p\|U - \bar{U}\|,$$

using integration, now based on estimate (39). For the $(1, 1)$ and $(1, 2)$ blocks we use directly (27) and (28) in Lemma 8. We only have to notice that for every interval $\mathcal{I} = [p_1, p_2]$ with $p_2 \le \epsilon_7$, picking $(U, p) \in \mathcal{N}_{p_1,p_2}$ implies $(x^+(U, p), U, p) \in \mathcal{W}(\epsilon_7)$ by (34), so that

$$\begin{aligned}
&\|U^+\left(x^+(U, p), U, p\right)_{11}\| + \|U^+\left(x^+(U, p), U, p\right)_{12}\| \\
&\quad \le K_4\left(p^2\|U - \bar{U}\| + \|x^+(U, p) - \bar{x}\|^2 + p\|U - \bar{U}\| + \|x^+(U, p) - \bar{x}\|\right) \\
&\quad \le 2K_4 p\|U - \bar{U}\|
\end{aligned}$$

by estimates (27) and (28), estimate (44) (a) with constant K_5, and the fact that we may render $1 + p + K_5^2 p\|U - \bar{U}\| + K_5^2\|U - \bar{U}\| \le 2$ by reducing $\epsilon_7 > 0$ if necessary. This takes into account that $U^+(U, p) = U^+(x^+(U, p), U, p)$. Altogether, we have shown $\|U^+(U, p) - \bar{U}\| \le (2K_4 + K_6(K_5 + 1))p\|U - \bar{U}\|$, proving the second part of estimate (44). If we put $K_7 = \max\{K_5, 2K_4 + K_6(K_5 + 1)\}$, we clearly obtain both estimates in (44) with the same constant K_7. ■

Remark. As a consequence of Lemma 13, we see that if we allow the penalty parameter p_k to shrink to 0, we obtain local superlinear convergence $U_k \to \bar{U}$, while $x_k \to \bar{x}$ converges R-superlinearly. See also [25] for a proof of this fact. Naturally, allowing the penalty parameter to converge to 0 leads to numerical ill-conditioning in the tangent program (5), and has to be avoided in practice. It is mandatory to freeze p at a decent positive value. During the following section, we show that the algorithm then still converges linearly if the initial U is sufficiently close to $\bar{U}$.

11. Progress measure

Recall that the progress measure $\sigma(x, U, p)$ used in step 4 of our algorithm is given as:

$$\sigma(x, U, p) = \|U - \left(I - p^{-1}G(x)\right)^{-1} U \left(I - p^{-1}G(x)\right)^{-1} \|. \tag{45}$$

Then in fact $\sigma(x^+, U, p) = \|U - U^+\|$. The test in step 4 therefore becomes $\|U - U^+\| \leq \tau \|U^- - U\|$, where $U = U^+(x, U^-, p^-)$. This is indeed a primal-dual progress test, because it takes the full information x, U, p from two consecutive sweeps into account.

LEMMA 14. *Suppose hypotheses* (H_1)–(H_3) *are satisfied at* $(\bar{x}, \bar{U})$. *Then there exists* $\epsilon_7 > 0$, $0 < \underline{p} < \overline{p} < \epsilon_7$, *a neighborhood N of* $\bar{U}$ *and a neighborhood* $\mathcal{U}$ *of* $\bar{x}$ *such that for all* p_1 *and* U_1 *satisfying* $\overline{p}\gamma < p_1 \leq \overline{p}$ *and* $U_1 \in N$:

1. *The sequences* p_k, U_k *and* $x_{k+1} = x^+(U_k, p_k)$ *generated by the augmented Lagrangian algorithm are well defined, and* $U_k \in N$ *for every k.*
2. $x_{k+1} \in \mathcal{U}$ *for all k.*
3. *The sequence* p_k *stays in the interval* $\mathcal{I} = [\underline{p}, \overline{p}]$, *and is therefore constant from some index* k_1 *on.*

Proof. Let ϵ_7 and the wedge neighborhood $\mathcal{W}(\epsilon_7)$ be as in the proof of Lemma 13. Choose $\overline{p} \leq \epsilon_7$ such that $K_7\overline{p} < 1$. For later use put

$$K_8 := K_3\epsilon_7 + \|I_{m-s}\| + K_3K_7\epsilon_7^2 + K_3K_7\epsilon_7^3,$$
$$K_9 := \|\bar{U}_{22}\|K_3^2K_7^2\epsilon_7^2 + 2K_3K_7\epsilon_7\|\bar{U}_{22}\| + K_3K_7$$

and define

$$\underline{p} := \min\left\{\gamma^2\overline{p}, \frac{\gamma^2\tau}{K_7\left(1 + K_8^2 + K_9 + \tau\right)}\right\} \tag{46}$$

where τ, γ are the parameters used in the algorithm, and where the constants K_i, ϵ_i are as in the previous sections. Recall that these constants are available before $\underline{p}$ is defined, because they have been collected as part of the prior information.

Now we define the neighborhoods in question by setting $\mathcal{U} = \mathcal{U}_{\underline{p},\overline{p}}$ and $\mathcal{N} = \mathcal{N}_{\underline{p},\overline{p}}$, $N = N_{\underline{p},\overline{p}}$. See section 10, formulas (33)–(36), for their definitions. Notice that by Lemma 11 we have $x_{k+1} = x^+(U_k, p_k)$ for the implicit function associated with $[\underline{p}, \overline{p}]$ for all k with $p_k \in \mathcal{I} = [\underline{p}, \overline{p}]$. In particular, the sequences x_{k+1}, U_k and p_k are well defined for these k. This is because U_{k+1} stays in the neighborhood $N_{\underline{p},\overline{p}}$ in view of estimate (46) (b) and $K_7\overline{p} < 1$, so that the procedure can be continued at the next step. In particular, from the uniqueness part of the implicit function theorem (33) we then know that x_{k+1} stays in $\mathcal{U} = \mathcal{U}_{\underline{p},\overline{p}}$ and is the unique local minimum (even unique critical point) of tangent program (5) in $\mathcal{U}$.

Suppose the sequence p_k does *not* stay in the interval $\mathcal{I} = [\underline{p}, \overline{p}]$. Then there exists a smallest index k_1 such that $p_{k_1} \in \mathcal{I}$, but $p_{k_1+1} = \gamma p_{k_1} < \underline{p}$. We will show that this leads to a contradiction.

Notice first that $p_{k_1} < p_1$, so that $k_1 \geq 2$. Indeed, $p_{k_1} = p_1 > \gamma \overline{p}$ would give $\underline{p} > p_{k_1+1} = \gamma p_{k_1} = \gamma p_1 > \gamma^2 \overline{p}$, contradicting the definition of $\underline{p}$. Hence indeed $k_1 \geq 2$.

Let $Z_k := (I - p_k^{-1} G(x_{k+1}))^{-1}$, where $x_{k+1} = x^+(U_k, p_k)$, $\bar{Z} = \operatorname{diag}(0_s, I_{m-s})$. Then $Z_k = Z_{p_k}(x^+(U_k, p_k))$. We have

$$\|U_k - \bar{U}\| \leq p_k \epsilon_7 \leq \overline{p}\epsilon_7 \leq \epsilon_7^2.$$

Using this, (26), and (44) (a) we have

$$\begin{aligned}
\|Z_k\| &\leq \|(Z_k)_{11}\| + \|(Z_k)_{22}\| + 2\|(Z_k)_{12}\| \\
&\leq \|(Z_k)_{11}\| + \|I_{m-s}\| + \|(Z_k)_{22} - I_{m-s}\| + 2\|(Z_k)_{12}\| \\
&\leq K_3 p_k + \|I_{m-s}\| + K_3 p_k^{-1}\|x_{k+1} - \bar{x}\| + 2K_3\|x_{k+1} - \bar{x}\| \\
&\leq K_3\epsilon_7 + \|I_{m-s}\| + K_3 K_7 \|U_k - \bar{U}\| + 2K_3 K_7 \epsilon_7 \|U_k - \bar{U}\| \\
&\leq K_3\epsilon_7 + \|I_{m-s}\| + K_3 K_7 \epsilon_7^2 + 2K_3 K_7 \epsilon_7^3 = K_8 \qquad (47)
\end{aligned}$$

according to the definition of K_8. Next consider the matrix expression

$$\bar{U} - Z_k \bar{U} Z_k = \begin{bmatrix} -Z_{12}^k \bar{U}_{22} Z_{12}^{k\top} & -Z_{12}^k \bar{U}_{22} Z_{22}^k \\ -Z_{22}^k \bar{U}_{22} Z_{12}^{k\top} & \bar{U}_{22} - Z_{22}^k \bar{U}_{22} Z_{22}^k \end{bmatrix}.$$

Using again (26), we have

$$\begin{aligned}
\|\bar{U} - Z_k \bar{U} Z_k\| &\leq \| \left(\bar{U} - Z_k \bar{U} Z_k\right)_{11} \| + 2\| \left(\bar{U} - Z_k \bar{U} Z_k\right)_{12} \| + \| \left(\bar{U} - Z_k \bar{U} Z_k\right)_{22} \| \\
&= \|Z_{12}^k \bar{U}_{22} Z_{12}^{k\top}\| + 2\|Z_{12}^k \bar{U}_{22} Z_{22}^k\| + \|\bar{U}_{22} - Z_{22}^k \bar{U}_{22} Z_{22}^k\| \\
&\leq \|\bar{U}_{22}\| \left(K_3 \|x_{k+1} - \bar{x}\|\right)^2 + 2K_3 \|\bar{U}_{22}\| \|x_{k+1} - \bar{x}\| + K_3 p_k^{-1} \|x_{k+1} - \bar{x}\| \\
&\leq \left(\|\bar{U}_{22}\| K_3^2 K_7^2 \epsilon_7^2 + 2K_3 K_7 \epsilon_7 \|\bar{U}_{22}\| + K_3 K_7\right) \|U_k - \bar{U}\| \\
&= K_9 \|U_k - \bar{U}\| \qquad (48)
\end{aligned}$$

using the definition of K_9. Combining (47) and (48) gives the estimate

$$\begin{aligned}
\sigma(x_{k+1}, U_k, p_k) &= \|U_k - Z_k U_k Z_k\| \\
&\leq \|U_k - \bar{U}\| + \|\bar{U} - Z_k \bar{U} Z_k\| + \|Z_k \bar{U} Z_k - Z_k U_k Z_k\| \\
&\leq \|U_k - \bar{U}\| + \|\bar{U} - Z_k \bar{U} Z_k\| + \|Z_k\|^2 \|U_k - \bar{U}\| \\
&\leq (1 + K_8^2 + K_9)\|U_k - \bar{U}\|. \qquad (49)
\end{aligned}$$

On the other hand, using estimate (44) (b) we have for $k \geq 2$:

$$\|U_k - \bar{U}\| \leq K_7 p_{k-1} \|U_{k-1} - \bar{U}\| \leq K_7 p_{k-1} \left(\|U_{k-1} - U_k\| + \|U_k - \bar{U}\|\right)$$

and therefore

$$\|U_k - \bar{U}\| \leq \left((K_7 p_{k-1})^{-1} - 1\right)^{-1} \|U_{k-1} - U_k\|, \qquad (50)$$

where we have $(K_7 p_{k-1})^{-1} > 1$ for all $k \geq 2$ by assumption. Combining (49) and (50) gives

$$\begin{aligned} \sigma(x_{k+1}, U_k, p_k) &\leq \frac{1 + K_8^2 + K_9}{(K_7 p_{k-1})^{-1} - 1} \|U_{k-1} - U_k\| \\ &= \frac{1 + K_8^2 + K_9}{(K_7 p_{k-1})^{-1} - 1} \sigma(x_k, U_{k-1}, p_{k-1}) \\ &=: \tau_k \, \sigma(x_k, U_{k-1}, p_{k-1}). \end{aligned} \tag{51}$$

Since the p_k are decreasing, the sequence τ_k defined by (51) decreases as well. Consequently, if we can find an index k_2 such that $\tau_{k_2} \leq \tau$, where τ is the parameter used in the algorithm, then we have $\tau_k \leq \tau$ for every $k \geq k_2$. According to step 4 of the algorithm, and due to (51), the parameter p_k would then be unchanged for $k \geq k_2$. In consequence, an index k_2 of this type could not possibly occur *before* k_1. Namely, suppose we had $k_2 \leq k_1$, then $p_k = p_{k_2}$ for $k \geq k_2$, contradicting the definition of k_1, where we have $p_{k_1+1} = \gamma p_{k_1}$.

What we therefore know is $k_2 > k_1$. In other words, $\tau_k > \tau$ for every $k < k_2$, meaning $\tau_k > \tau$ for every $k \leq k_1$. In particular $\tau_{k_1} > \tau$. Setting $K_{10} := 1 + K_8^2 + K_9$, this becomes

$$\tau_{k_1} = \frac{K_{10}}{(K_7 p_{k_1-1})^{-1} - 1} > \tau$$

if we plug in the expression (51) for τ_{k_1}. This is now the same as

$$p_{k_1-1} > \frac{\tau}{K_7(K_{10} + \tau)}.$$

Since $p^+ \in \{p, \gamma p\}$ at each step of the algorithm, we deduce

$$p_{k_1} \geq \gamma p_{k_1-1} > \frac{\gamma \tau}{K_7(K_{10} + \tau)}.$$

Using $p_{k_1+1} = \gamma p_{k_1}$ then gives

$$p_{k_1+1} > \frac{\gamma^2 \tau}{K_7(K_{10} + \tau)}.$$

On the other hand, $p_{k_1+1} < \underline{p}$ by construction, which means

$$\underline{p} > \frac{\gamma^2 \tau}{K_7(K_{10} + \tau)} = \frac{\gamma^2 \tau}{K_7(1 + K_8^2 + K_9 + \tau)}.$$

This contradicts the definition (46) of $\underline{p}$. ■

Remark. If the sequences x_k, U_k and p_k generated by the augmented Lagrangian algorithm are started with initial $p_1 > \overline{p}$, then, as the p_k are reduced, there will be a smallest k_0 with $p_{k_0} \leq \overline{p}$. Then $p_{k_0} > \gamma \overline{p}$, and the conclusions of Lemma 14 are still valid for the sequence (U_k, p_k), $k \geq k_0$, if $U_{k_0} \in \mathcal{U}$ for the neighborhood $\mathcal{U}$ found in Lemma 14. Naturally, as we prove a local convergence result, such a restriction has to be expected.

Remark. The initial parameter p_1 must fall in the range $(\gamma \overline{p}, \overline{p}]$, which appears small. However, since $p^+ \in \{\gamma p, p\}$ at each step, some p_k always falls within this range as the p_k, starting large, get smaller. On the other hand, if we feel uncomfortable with this initial condition, we can easily replace it by $p_1 \in (\gamma^a \overline{p}, \overline{p}]$ for some large $a > 1$, so that $\gamma^a \overline{p} \ll \overline{p}$, by adapting the definition (46) of $\underline{p}$ in the proof.

Assembling the findings of the previous sections leads to the following local convergence theorem.

THEOREM 1. *Let $\bar{x}$ be a local minimum of* (1) *with associated Lagrange multiplier $\bar{U}$ such that the hypotheses (H_1)–(H_3) are satisfied. Then there exists a neighborhood N of $\bar{U}$, a neighborhood $\mathcal{U}$ of $\bar{x}$, and $\overline{p} > 0$ such that the following conditions are satisfied:*

1. *Whenever $U_1 \in N$ and $\gamma \overline{p} < p_1 \le \overline{p}$, then the sequences U_k, $p_k > 0$ and x_{k+1} generated by the augmented Lagrangian algorithm are well-defined if x_{k+1} is the local minimum of $\min_{x \in \mathbb{R}^n} F(x, U_k, p_k)$ in $\mathcal{U}$. The sequence U_k stays in N, and x_{k+1} is the unique critical point of* (5) *in $\mathcal{U}$.*
2. *The sequence U_k converges to $\bar{U}$ with Q-linear speed, and x_k converges to $\bar{x}$ with R-linear speed.*
3. *The sequence $p_k > 0$ is constant from some index k_1 on.*

Proof. We choose $\overline{p}$, $\underline{p}$, and then $\mathcal{U}$ and N as in the proof of Lemma 14. Then we know that the sequence p_k does not leave the interval $[\underline{p}, \overline{p}]$. Since it is decreasing, it is eventually constant with value $\hat{p} \in [\underline{p}, \overline{p}]$.

Now $x_{k+1} = x^+(U_k, \hat{p})$ and $U_{k+1} = U^+(U_k, \hat{p})$, so estimate (44) (b) immediately shows that U_k converges Q-linearly with speed $K_7 \hat{p} \le K_7 \overline{p} < 1$. By (44) (a), x_k then converges R-linearly. ■

12. Example

Let us indicate by way of an example that condition (13) is too strong in general. Consider the program

$$\text{minimize } f(x) = \frac{1}{2}\left(-x_1^2 - x_2^2\right)$$
$$\text{subject to } G(x) = \begin{bmatrix} -1 & 1 - x_1 & 0 \\ 1 - x_1 & -1 & -x_2 \\ 0 & -x_2 & -1 \end{bmatrix} \preceq 0$$

whose unique minimum is $\bar{x} = (2, 0)$. The Hessian of the Lagrangian is $L_{xx}(x, U) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, because $G'' = 0$. That already means that (13) has no chance to be true. Observe that $G'(\bar{x})^\star U = (-2u_{12}, -2u_{23})$, so that the KKT-conditions read

$$\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2u_{12} \\ -2u_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

along with complementarity, which gives

$$\bar{U} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix Q which diagonalizes $G(\bar{x})$ and $\bar{U}$ in the sense that $\bar{U} = Q\text{diag}\bar{U}Q^\top$ is

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The curvature term is therefore

$$\mathcal{H}(\bar{x}, \bar{U}) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

The final link is obtained by computing the critical cone. According to (10), we obtain $C(\bar{x}) = \mathbb{R}(0, 1)$ here. And it can indeed be verified that

$$h^\top \left(L_{xx}(\bar{x}, \bar{U}) + \mathcal{H}(\bar{x}, \bar{U})\right) h = [0\ h_2] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ h_2 \end{bmatrix} = h_2^2 > 0$$

whenever $h \in C(\bar{x}),\ h \neq 0$. That means the second-order no-gap sufficient optimality condition (11) is satisfied, even though $L_{xx}(\bar{x}, \bar{U}) \prec 0$.

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