Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

ON THE THEORY OF B- AND B_r-SPACES IN GENERAL TOPOLOGY

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(Received August 5, 1986)

1. ***B*- and *B*$_r$-spaces.** A T_2 topological space E is called a *B_r-space* (*B-space*) if every continuous, nearly open bijection (surjection) f from E onto an arbitrary T_2 space F is open. Here $f: E \to F$ is called *nearly open* if for every $x \in E$ and every neighbourhood U of x the set $\mathrm{cl}(f(U))$ is a neighbourhood of $f(x)$.

The notions of B- and B_r-spaces in the above sense have first been used by T. Husain in the categories of locally convex vector spaces ([Hu$_1$]) and topological groups ([Hu$_2$]). They have been chosen in reminiscence of V. Pták's open mapping theorems ([P], [Kö]). We have adopted Husain's definition for the topological case. References concerning the classical theory of B- and B_r-spaces and groups are [P], [Kö], [AEK], [Hu$_i$], [Ba$_i$], [Pe], [Gr], [Su], etc. In a purely topological context, B_r-spaces have been considered in [We], [BP], although the term 'B_r-space' has not been used there. Further references are [Wi], [St], [N$_i$].

Every T_2 locally compact space is a B-space and every B-space is a B_r-space. In [We], Weston proved that every completely metrizable space is a B_r-space. In [BP] this has been generalized to Čech complete spaces. In [N$_1$] we have further generalized this to obtain.

Proposition 1. *Every T_2 semi-regular topological space E containing a dense Čech complete subspace is a B_r-space. In particular, this is true for monotonically Čech complete spaces.*

In [N$_1$] we have given a direct proof. Proposition 1 may also be deduced from Byczkowski and Pols' result [BP] if we use the following

Lemma. *Let E be a T_2 semi-regular space and let F be a T_2 space. Let $f: E \to F$ be a continuous, nearly open bijection and suppose there exists a dense subset D of E such that $f \mid D: D \to f(D)$ is open. Then f is open.*

Proof. Let $x \in E$ and a neighbourhood U of x be fixed. Choose a regular-open neighbourhood V of x contained in U. We prove $\mathrm{int\,cl}(f(V)) \subset f(U)$. Let $z \in \mathrm{int\,cl}(f(V))$, $z = f(y)$. Let W be a neighbourhood of y with $f(W) \subset \mathrm{int\,cl}(f(V))$. It is sufficient to prove $W \subset \bar{V}$. So let $w \in W$ and let O be a regular-open neighbourhood of w contained in W. Proving that $O \cap V \neq \emptyset$ remains.

Since O, V are regular-open in E, $O \cap D$, $V \cap D$ are regular-open in D, hence $f(O \cap D), f(V \cap D)$ are regular-open in $f(D)$. But note that $\mathrm{int\,cl}(f(O)) \cap f(D)$ and $\mathrm{int\,cl}(f(V)) \cap f(D)$ are as well regular-open in $f(D)$ and this implies $\mathrm{int\,cl}(f(O)) \cap \cap f(D) = f(O) \cap f(D)$, $\mathrm{int\ cl}(f(V)) \cap f(D) = f(V) \cap f(D)$. Since $O \subset W$ implies $\mathrm{int\,cl}(f(O)) \subset \mathrm{int\,cl}(f(V))$ we obtain the desired result $O \cap V \neq \emptyset$. □

In [N_3] we have investigated an interesting class of B-spaces.

Proposition 2. *Every Lindelöf P-space is a B-space.* □

Using the lemma above, one may obtain the following result. Here 'locally Lindelöf' means that every point has a base of neighbourhoods consisting of Lindelöf subspaces.

Proposition 3. *Every T_2 semi-regular locally Lindelöf space E containing a dense set of P-points is a B_r-space.*

Proof. Let $f: E \to F$ be a continuous, nearly open bijection onto the T_2 space F. We may assume that F is semi-regular. Let D denote the set of P-points in E. We prove that $f \mid D: D \to f(D)$ is open. First note that every point of $f(D)$ is a P-point in F. Indeed, let G_n, $n = 1, 2, \ldots$ be open sets containing $y = f(x)$, $x \in D$. Choose open sets V_n, $n = 1, 2, \ldots$ in E having $x \in V_n$, $\mathrm{int\,cl}(f(V_n)) \subset G_n$. Then $V = \bigcap_n V_n$ is a neighbourhood of x having $\mathrm{int\,cl}(f(V)) \subset G_n$, $n = 1, 2, \ldots$.

Let $x \in D$ and a Lindelöf neighbourhood U of x be fixed. We claim that $\mathrm{cl}(f(U)) \cap \cap f(D) = f(U) \cap f(D)$. Assume the contrary and let $z \in \mathrm{cl}(f(U)) \setminus f(U)$, $z = f(y)$, $y \in D$. Let Φ denote the filter of neighbourhoods of z, then $\{f(U) \setminus \bar{O} : O \in \Phi\}$ is an open cover of $f(U)$, hence there exist $O_n \in \Phi$, $n = 1, 2, \ldots$ having $f(U) = \bigcup_n f(U) \setminus O_n$, a contradiction since we have $\bigcap_n O_n \in \Phi$. □

It follows from our lemma that every T_2 semi-regular space E containing a dense B_r-subspace is itself a B_r-space. The corresponding result for B-spaces is not valid. In § 7 we shall present an example of a completely regular space E containing a dense Lindelöf P-subspace which is not a B-space.

In [N_2] we have investigated another interesting class of B_r-spaces. Let S be a cofinal subset of ω_1. Let S^* denote the set of $f \in \omega_1^\omega$ having $f^* = \sup\{f(n): n < \omega\} \in \in S$. Give ω_1 the discrete topology and let ω_1^ω and S^* have the product topology. Recall that S is called *stationary* if it intersects every closed cofinal subset of ω_1. We have the following

Proposition 4. ([N_2], [FK] for (1) $\Leftrightarrow$ (2)). *Let $S \subset \omega_1$ be cofinal. Then the following statements are equivalent*:

(1) *S is stationary*;
(2) *S^* is a Baire space*;
(3) *S^* is a B_r-space.* □

This provides examples of metrizable B_r-spaces which do not contain any dense completely metrizable subspace, since clearly S^* contains a dense completely metrizable subspace if and only if S contains a closed cofinal subset.

2. Order interpretation. We introduce an order relation $\leqq$ on the set of all T_2 topologies on a fixed set E by postulating that $\tau_1 \leqq \tau_2$ is satisfied if and only if id: $(E, \tau_2) \to (E, \tau_1)$ is continuous and nearly open. Then (E, τ) is a B_r-space if and only if τ is minimal among T_2 topologies on E. Dually one may consider the $\leqq$ maximal topologies. It turns out that these can be internally characterized as follows.

Proposition 5. *τ is maximal with respect to $\leqq$ if and only if every dense subset of (E, τ) is open.* □

Open problem. *Obtain an internal characterization of $\leqq$ minimal* (i.e. B_r-) *topologies.*

Using the Kuratowski/Zorn lemma one easily proves that given any T_2 topology τ on E, there exists $a \leqq$ maximal topology τ_0 having $\tau \subset \tau_0$.

Open problem. *Does a corresponding result hold for $\leqq$ minimality?*

3. Category. Since T_2 minimal ($= H$ minimal) topological spaces are clearly B_r-spaces, it follows from a result of Herrlich ([He]) that a B_r-space need not be a Baire space in general. One may ask, however, for a first category B_r-space which is completely regular. In [N_3] we have provided an example of this type constructing a first category Lindelöf P-space. On the other hand, all metrizable B_r-spaces known up to now are Baire spaces. In [N_3] we have obtained the following

Theorem 1. *Every strongly zero-dimensional metrizable B_r-space is Baire.* □

Open problem. *Is it true that every metrizable B_r-space is a Baire space?*

Note that theorem 1 may be used to prove that every suborderable metrizable B_r-space is a Baire space. Another partial positive answer is obtained for metrizable topological groups in view of the following

Proposition 6. ([N_2]) *Every topological group which is a B_r-space (in the topological sense) is complete with respect to its two-sided uniformity.* □

4. Products. The situation in the classical categories (see [Kö], [Gr]) suggests that the product of even two B_r-spaces need not be a B_r-space. In [N_2] we have obtained the expected counterexamples.

Proposition 7. *Let $S, T \subset \omega_1$ be stationary sets. Then the following are equivalent:*

(1) *$S \cap T$ is stationary;*

(2) *$S^* \times T^*$ is a B_r-space.* □

Clearly this provides the desired counterexamples for we may choose disjoint stationary subsets S, T of ω_1, then S^*, T^* are B_r-spaces, but $S^* \times T^*$ is not.

One may ask for a B_r-space E whose square $E \times E$ is no longer a B_r-space. Such an example can be obtained from the following construction.

Proposition 8. *Let F be a strongly zero-dimensional metrizable Baire space such that for some $n \geqq 2$ F^n is no longer a Baire space. Suppose that F is a B_r-space. Then there exists r, $1 \leqq r \leqq n-1$ such that $E = F^r$ is a B_r-space but $E \times E$ is not.*

Proof. The construction is based on theorem 1 and the fact that finite products of strongly zero-dimensional metrizable spaces are strongly zero-dimensional and metrizable. Regard $F \times F$. If this is not a B_r-space, then $E = F$. Otherwise F^2 is a Baire space by theorem 1. Then regard $F^2 \times F^2$. If this is not B_r, then $E = F^2$. Otherwise F^4 is a Baire space. etc. □

In [N_3] we have obtained a space F as above using an example from [FK].

Though no general positive results concerning products of B_r-spaces are to be expected, there are positive results in special situations. Namely the classes of T_2 minimal spaces, Čech complete spaces, Lindelöf P-spaces are examples of productive, countably productive, finitely productive classes of B_r-spaces.

Open problem. *Given a B_r-space E and a compact T_2 space K, must $E \times K$ be a B_r space?*

5. Closed subspaces. From the situation in the classical categories (concerning the open mapping theory) one would expect that closed subspaces of B_r-spaces are again B_r. In fact, the corresponding statements are known to be valid in the categories of locally convex vector spaces ([Kö]), linear topological spaces ([AEK]) and Abelian topological groups. In the case of topological groups the answer is not known (see [Ba_2], [Gr]) although there are some positive partial results. In the topological case, the situation seems to be of a completely different nature for we have the

Proposition 9. *Every T_2 semi-regular topological space E is the closed subspace of some B_r-space F.*

Proof. Let $F = E \times \{1\} \cup E \times \{2\}$ and define a topology on F by imposing that $\{(x, 1)\}$ is a neighbourhood of $(x, 1)$ for each $x \in E$ and $U(x)$ is a neighbourhood of $(x, 2)$, whenever $x \in E$ and U is a neighbourhood of x in E, where $U(x)$ denotes the set $\{(y, i): y \in U \setminus \{x\},\ i = 1, 2\} \cup \{(x, 2)\}$. Then $E \times \{2\}$ is a closed subspace of F homeomorphic with E and $E \times \{1\}$ is an open dense and discrete subspace of F. Since F is semi-regular by construction, it is a B_r-space by proposition 1. □

6. Sums of B_r-spaces. The class of B_r-spaces behaves very strange with respect to summation. First note that the sum of even two B_r-spaces need not be a B_r-space. Indeed, let S, T be disjoint stationary subsets of ω_1, then S^*, T^* are B_r-spaces but $S^* + T^*$ is not B_r in view of the fact that S^*, T^* are disjoint dense subspace of ω_1^ω and hence the natural mapping $f\colon S^* + T^* \to \omega_1^\omega$ is a continuous nearly open bijection onto $f(S^* + T^*)$ which is not open.

On the other hand there are certain positive results on sums of B_r-spaces.

Proposition 10. ([N_2]) *Given any B_r-space E, the sum $E + E$ is a B_r-space.* □

In [N_2] we have investigated summation with Čech complete summands and have obtained the following interesting

Theorem 2. *Let E be a completely regular B_r-space. Then the following statements are equivalent:*
(1) *E is a Baire space;*
(2) *$E + F$ is a B_r-space whenever F is Čech complete.* □

As a consequence of theorem 1 and theorem 2 we deduce that $E + F$ is a B_r-space if E is a strongly zero-dimensional metrizable B_r-space and F is Čech complete. On the other hand, if E is a Lindelöf P-space of the first category, theorem 2 provides a Čech complete space F such that $E + F$ is no longer a B_r-space.

Another positive result on sums is the following

Proposition 11. *Given a B_r-space E and a T_2 locally compact space L, the sum $E + L$ is a B_r-space.*

Proof. Let $f: E + L \to F$ be a continuous, nearly open bijection onto the T_2 space F. Since $f \mid E: E \to f(E)$, $f \mid L: L \to f(L)$ are as well nearly open, we have $E \simeq f(E)$, $L \simeq f(L)$. It remains to prove that $f(E)$ is closed in F. But this follows from the fact that $f(L)$ is open in its T_2 extension int $\mathrm{cl}(f(L))$ and so is open in F. □

7. *B*-spaces. It has been an open question for a long time whether there exist B_r-complete locally convex vector spaces which are not B-complete. Finally, an example of this type has been found by Valdivia ([V]). In the category of topological groups the corresponding counterexample was constructed in [Su]. Now in the purely topological case the situation is different. While the class of B_r-spaces is considerably large, B-spaces seem to be of a rather special type. In fact, even completely metrizable spaces need not be B-spaces. An example may be found in [BP].

Example. A T_2 minimal space need not be a B-space. Indeed, let E denote the T_2 minimal space constructed in [He], whose point set is $R_0 \cup R_1 \cup R_2$, where $R_0 = \mathbf{R} \setminus \mathbf{Q} \cap I \times \{0\}$, $R_i = \mathbf{Q} \cap I \times \{i\}$, $i = 1, 2$. Define $f: E \to I$ by $f(x, i) = x$, then f is a continuous, nearly open surjection which is not open.

Concerning sums of B-spaces we have the following

Proposition 12. ([N_3]). *Let E be a completely regular B-space. Then the following statements are equivalent:*
(1) *$E + L$ is a B-space whenever L is T_2 locally compact;*
(2) *$E + K$ is a B-space whenever K is T_2 compact;*
(3) *$E + \beta E$ is a B-space;*
(4) *E is locally compact.* □

Let E be a non-discrete Lindelöf P-space. Then E is a B-space but $E + \beta E$ is not since E is not locally compact. On the other hand, $E + E$ is clearly a B-space since it is Lindelöf P. This proves that the lemma from § 1 is not valid for surjective mappings f resp. the class of B-spaces is not closed with respect to taking T_2 extensions.

Bibliography

[AEK] *N. Adasch, B. Ernst, D. Keim:* Topological vector spaces LNM 639.

[Ba_1] *J. W. Baker:* Inversion theorems and compactness. JLMS *47* (1976), 409ff.

[Ba_2] *J. W. Baker:* On a generalized open mapping theorem. MA *172* (1976), 217ff.

[BP] *T. Byczkowski, R. Pol.:* On the closed and open mapping theorems. Bull. Acad. Pol. Sci. *24* (1976), 723—726.

[FK] *W. G. Fleissner, K. Kunen:* Barely Baire spaces. FM *CI* (1978), 229—240.

[Gr] *D. L. Grant:* Topological groups which satisfy an open mapping theorem. PJM *68* (1977), 411—423.

[He] *H. Herrlich:* Nicht alle T_2 minimalen Räume sind von zweiter Kategorie. MZ *91* (1966), 15.

[Hu_1] *T. Husain:* The open mapping and closed graph theorem in topological vector spaces. Braunschweig 1965.

[Hu_2] *T. Husain:* Introduction to topological groups. Philadelphia 1966.

[Kö] *G. Köthe:* Topological vector spaces II. New York 1979.

[N_1] *D. Noll:* Open mapping theorems in topological spaces. CMJ *35* (110), 1985.

[N_2] *D. Noll:* Sums and products of B_r-spaces. PJM *123*, 1, 1986.

[N_3] *D. Noll:* Baire category and B_r-spaces. PAMS *101*, 671 ff, 1987.

[Pe] *B. J. Pettis:* On continuity and openness of homomorphisms in topological groups. Ann. of Math. *52* (1957), 293—308.

[P] *V. Pták:* Completeness and the open mapping theorem. Bull. Soc. Math. France *86*, 41—74, 1958.

[St] *J. D. Stein:* The open mapping theorem for spaces with unique segments. PJM *40* (*2*), 1972, 459—462.

[Su] *L. J. Sulley:* A note on B- and B_r-complete topological Abelian groups. Proc. Camb. Phil. Soc. *66* (1969), 275.

[We] *J. D. Weston:* On the comparison of topologies. JLMS *32* (1957), 342—354.

[Wi] *M. Wilhelm:* Criteria for the openness of relations. FM *CXIV* ('81), 219ff.

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