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OPEN MAPPING THEOREMS IN TOPOLOGICAL SPACES

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Introduction. Open mapping theorems for topological spaces have been proved by several authors ([We], [BP], [Wi]) in the following form:

Let f be a continuous, nearly open bijection (surjection) from a topological space E to a topological space F. Then, under appropriate conditions on E and F, f will be open.

In [We], Weston has proved that if E is assumed to be a completely metrizable space, F a Hausdorff space and if f is bijective, then the open mapping theorem holds. Byczkowski and Pol ([BP]) have extended Weston's result to Čech-complete spaces E. Here we prove that E may be supposed to be a semi-regular space which densely contains the open continuous image of some paracompact Čech-complete space (see section 2). Moreover we prove that the open mapping theorem also holds if E is a semi-regular space which is pseudo-complete in the sense of Oxtoby and F is assumed to have a G_δ-diagonal (see section 3).

The second part of our paper is devoted to the study of some applications in the classes of semitopological and topological groups. In section 5 we introduce a notion of 'barrelledness' for semitopological and topological groups. The latter is then used to establish a Banach-Steinhaus theorem for 'barrelled' topological groups, by reducing it to an open mapping theorem.

Although the discussion of open mapping theorems is usually combined with that of graph theorems, we do not prove any graph theorems here. Rather hope we to deal with this topic in a subsequent paper.

0. Preliminaries. In this section we summarize a few technical notions in order to simplify the reading of the text. Note that our topological language is adopted from Engelking's book [E].

0.1. Nearly open and nearly continuous mappings. Let E, F be topological spaces. A mapping f from E to F is called *nearly open* if for every $x \in E$ and every neighborhood U of x, the set $\mathrm{cl}\,(f(U))$ is a neighborhood of $f(x)$. Dually, f is called *nearly continuous* if for every $x \in E$ and every neighborhood V of $f(x)$ the set $\mathrm{cl}\,(f^{-1}(V))$ is a neighborhood of x. (Compare [Kö], p. 24 and p. 36, [Wi], [We], [BP]).

0.2. Webs. Let E be a topological space. A pair (ϕ, T), consisting of a tree $T = (T, \leqq_T)$ ([KM], p. 84) of height ω and a mapping ϕ with domain T is called a *web* on E if the following two conditions are fulfilled:

(i) The set $\{\phi(t): t \in T\}$ is a pseudo-base for E (i.e. every nonempty open U in E contains some nonempty $\phi(t)$ and all $\phi(t)$, $t \in T$ are open sets).

(ii) Whenever $t \in T$ then $\{\phi(s): t <_T s \in T\}$ is a pseudo-base for the subspace $\phi(t)$.

A web (ϕ, T) on E is called *strict* if (i) and (ii) above hold with the term 'pseudo-base' replaced by the term 'base'. ■

(The reader might consult [CČN] here for the notion of a 'sieve', which is quite similar with that of a web).

0.3. Sets of interior condensation. The concept of a set of interior condensation has been introduced by Wicke and Worrell (see [WW]). We reproduce their definition in a slightly modified form.

A subset P of a topological space E is called a *set of interior condensation in* E if there exists a pair (ϕ, T), consisting of a tree T of height ω and a mapping ϕ from T to the topology of E, such that the following hold true:

(i) $\{\phi(t): t \in T\}$ is a cover of P.

(ii) Whenever $t \in T$, then $\{\phi(s): t <_T s \in T\}$ covers $P \cap \phi(t)$.

(iii) If $b \subseteq T$ is a cofinal branch ([KM], p. 84), then $\bigcap\{\phi(t): t \in b\} \subseteq P$. ■

Note that every G_δ-set is a set of interior condensation. The converse, however, is not true in general. (See [WW]).

0.4. σ-discrete decomposability. In [Ha], Hansell has introduced σ-discretely decomposable sets of subsets of a topological space E. A set $\{X_i: i \in I\}$ of subsets of E is called *σ-discretely decomposable* (abb. σ-d.d.) if there exist sets $X_{i,n}$, $n \in \mathbb{N}$, $i \in I$ such that for every $n \in \mathbb{N}$ the set $\{X_{i,n}: i \in I\}$ is discrete ([E], p. 33), and for every $i \in I$ we have $X_i = \bigcup\{X_{i,n}: n \in \mathbb{N}\}$. Furthermore, Hansell calls σ-discrete f from a metrizable space E to a metrizable space F if there exists a σ-d.d. set $\mathscr{R}$ of subsets of E such that for every open $V \subseteq F$ we have $f^{-1}(V) = \bigcup\{B \in \mathscr{R}: B \subseteq f^{-1}(V)\}$. (See [Ha]).

0.5. The operator D. Let E be a topological space. If X is a subset of E, then $D(X)$ denotes the set of all $x \in E$ such that X is of the second category relative to x. For the most important properties of D we refer the reader to [KM], p. 428ff. Here it will be sufficient to know that $D(X)$ is always closed and contained in cl X, and that $X \setminus D(X)$ is always of the first category in E.

1. Complete spaces. In this section we recall two concepts of completeness in topological spaces which belong to the wide field of 'strong Baire properties'. (See for example [AL] for related concepts).

Definition 1. A topological space E is called *c-complete* if there exists a web (ϕ, T) on E with the following property:

(c) Whenever $b \subseteq T$ is a cofinal branch and $\mathscr{F}$ is a filter on E with $\phi(t) \in \mathscr{F}$ for all $t \in b$, then $\mathscr{F}$ has a cluster point in $\bigcap\{\phi(t): t \in b\}$.

If, in addition, the web (ϕ, T) is strict, then E is called *strictly* c-complete. ■

In the frame of regular hausdorff spaces, the strictly c-complete spaces are known under the name 'monotonically Čech-complete spaces', used in [CČN], and as 'spaces with condition $\mathscr{K}$', used in [WW]. In [WW] spaces with condition $\mathscr{K}$ are characterized as those spaces which are open continuous images of paracompact Čech-complete spaces. A slight modification of the proof given there shows that this characterization remains valid for strictly c-complete spaces. In the completely regular case the strictly c-complete spaces are precisely those spaces which are sets of interior condensation in their Stone-Čech compactification (compare [WW]).

The c-complete spaces may in turn be characterized, in the frame of hausdorff spaces, as those spaces which densely contain a Čech-complete subspace or, equivalently, as those spaces which densely contain some strictly c-complete space. (For the ideas of proof compare [AL]).

Definition 2. A topological space E is called *p-complete* if there exists a web (ϕ, T) on E with the following property:

(p) Whenever $b \subseteq T$ is a cofinal branch with $\phi(t) \neq \emptyset$ for all $t \in b$, then also $\bigcap\{\phi(t): t \in b\} \neq \emptyset$.

If, in addition, the web (ϕ, T) is strict, then E is called *strictly p-complete.* ■

(Strict) c-completeness obviously implies (strict) p-completeness. The class of p-complete spaces is already known under the name 'weakly α-favourable spaces', introduced by White ([Wh]). However we prefer the name 'p-complete', since it indicates a close relation to the class of pseudo-complete spaces, introduced by Oxtoby (compare [Ox] and [AL]).

2. First open mapping theorem. In this section we prove an open mapping theorem which generalizes theorems in [We] and [BP], [Wi]. Moreover, there are similar open mapping theorems in [BP] and [Wi] which are very close to our theorem (see remark 4 below). Let us, however, first recall that a topological space E is called *semi-regular* if it has a base consisting of regular-open sets (also: open domains, see [E], p. 37).

Theorem 1. *Let E be a semi-regular c-complete space. Let f be a continuous, nearly open bijection from E to some hausdorff topological space F. Then f is an open mapping.*

Proof. Let $x \in E$ and a neighborhood U of x be fixed. By the semi-regularity of E there exists an open V such that $x \in V$ and $\operatorname{int} \overline{V} \subseteq U$. The proof will be finished if we will have proved $\operatorname{int}(\operatorname{cl}(f(V))) \subseteq f(U)$. To this end let $y \in \operatorname{int}(\operatorname{cl}(f(V)))$. Let $z \in E$ be chosen with $f(z) = y$. It will be sufficient to prove $z \in \operatorname{int}(\operatorname{cl} V)$. By the continuity of f there exists an open set O with $z \in O$ and $f(O) \subseteq \operatorname{int}(\operatorname{cl}(f(V)))$. If we can show that $O \subseteq \mathrm{cl}\, V$, the proof will be finished. Now take $b_0 \in O$. Fix an open neighborhood W of b_0 with $W \subseteq O$. We have to prove $V \cap W \neq \emptyset$.

Let (ϕ, T) be a web on E with condition (c).

By condition (i) in 0.2 we have $\mathrm{cl}(f(V)) = \mathrm{cl}(f(\bigcup\{\phi(t): t \in T,\ \phi(t) \subseteq V\}))$. But $f(b_0) \in f(O) \subseteq \mathrm{int}(\mathrm{cl}(f(0))) \subseteq \mathrm{int}(\mathrm{cl}(f(V)))$ and $f(b_0) \in \mathrm{int}(\mathrm{cl}(f(W)))$. Therefore we have $\mathrm{int}(\mathrm{cl}(f(W))) \cap f(\bigcup\{\phi(t): t \in T,\ \phi(t) \subseteq V\}) \neq \emptyset$. Choose $t_1 \in T$ with $\phi(t_1) \subseteq V$ and $a_1 \in \phi(t_1)$ with $f(a_1) \in \mathrm{int}(\mathrm{cl}(f(W)))$. Next we have $\mathrm{cl}(f(W)) = \mathrm{cl}(f(\bigcup\{\phi(s): s \in T,\ \phi(s) \subseteq W\}))$, again by (i) in 0.2. Since now $f(a_1) \in \mathrm{int}(\mathrm{cl}(f(\phi(t_1))))$, by the nearly openness of f we have $\mathrm{int}(\mathrm{cl}(f(\phi(t_1)))) \cap f(\bigcup\{\phi(s): s \in T,\ \phi(s) \subseteq W\}) \neq \emptyset$. Choose $s_1 \in T$ with $\phi(s_1) \subseteq W$ and $b_1 \in \phi(s_1)$ such that $f(b_1) \in \mathrm{int}(\mathrm{cl}(f(\phi(t_1))))$.

Proceeding in this way we inductively define sequence $(a_n)_{n\in N}$, $(b_n)_{n\in N}$, $(t_n)_{n\in N}$, $(s_n)_{n\in N}$ such that the following hold true for all $n \geqq 2$:

(α) $t_{n-1} <_T t_n$, $a_n \in \phi(t_n)$, $f(a_n) \in \mathrm{int}(\mathrm{cl}(f(\phi(s_{n-1}))))$;

(β) $s_{n-1} <_T s_n$, $b_n \in \phi(s_n)$, $f(b_n) \in \mathrm{int}(\mathrm{cl}(f(\phi(t_n))))$.

Now, by condition (c) of the web, the sequence (b_n) has a cluster point $b \in \bigcap\{\phi(t_n): n \in \mathbb{N}\} \subseteq W$. Hence the sequence $(f(b_n))$ in F has cluster point $f(b)$. Now for every open neighborhood G of $f(b)$ there exist natural numbers $n(G, 1) < n(G, 2) < \ldots$ such that $f(b_{n((G,i)})) \in G$ for all $i \in \mathbb{N}$. Hence, by (β), we find elements $c(G, i)$ of $\phi(t_{n(G,i)})$ such that $f(c(G, i)) \in G$ for all $i \in \mathbb{N}$. Now let $\mathscr{L}$ be the directed set of all pairs (G, i), where the order is defined in the natural way. Then, by (c), the net $\mathscr{N}$ of all $c(G, i)$ with $(G, i) \in \mathscr{L}$ has a cluster point $c \in \bigcap\{\phi(t_n): n \in \mathbb{N}\} \subseteq V$. But, obviously, the net $f(\mathscr{N})$ converges to $f(b)$ and, F being a hausdorff space, this yields $f(c) = f(b)$. This in turn gives $c = b$ for f is assumed to be injective. But now we have established $V \cap W \neq \emptyset$. ■

Remark 1. The assumption of semi-regularity on the space E cannot be relaxed here, as indicates the following proposition.

Let E be a hausdorff topological space, such that every continuous, nearly open bijection f from E to an arbitrary hausdorff space F is open. Then E must be semi-regular.

Proof. The set of all regular-open subsets of E is a base for a new topology on E. E, with this new topology, call F. Then the identic mapping f from E to F is bijective, continuous and nearly open. Since F is again a hausdorff space, f is open. But then E has the desired base. ■

Remark 2. It is clear that in theorem 1 the bijectivity of the mapping f cannot be replaced by injectivity. We can even say a little more:

Let E be a completely regular topological space. The following conditions are equivalent:

(1) *Every continuous, nearly open and dense injection f from E to an arbitrary hausdorff space F is open.*

(2) *E is locally compact.*

Proof. (1), applied to the embedding $E \to \beta E$, yields the opennes of E in βE. On the other hand, if U is a compact neighborhood of some $x \in E$, then $\operatorname{cl}(f(U)) = f(U)$, which proves the openness of f. ■

Remark 3. In [BP] is given an example for the fact that theorem 1 does not remain true if bijectivity of f is replaced by surjectivity. However, the situation is not so hopeless as this example might suggest.

Theorem 2. *Let E be a semi-regular c-complete space and let f be a continuous, nearly open surjection from E onto a hausdorff semi-regular space F. Then f is an extremal epimorphism* (see [HS], p. 110) *in the category of semi-regular topological spaces and continuous mappings.*

Proof. Let f be factorized to $h \circ g$ where $g: E \to G$, $h: G \to F$, G being a semi-regular space and h being a monomorphism (see [HS], p. 38). h must therefore be bijective. We have to prove that it is a homeomorphism. Let $x' \in G$ and some open neighborhood U' of x' be fixed. By semi-regularity, choose an open neighborhood V' of x' with $\operatorname{int}(\operatorname{cl} V') \subseteq U'$. We prove that $\operatorname{int}(\operatorname{cl}(h(V'))) \subseteq h(U')$. Take $y \in \operatorname{int}(\operatorname{cl}(h(V')))$. Choose $z \in E$ with $f(z) = y$. Let O' be a neighborhood of $g(z)$ in G with $h(O') \subseteq \operatorname{int}(\operatorname{cl}(h(V')))$. The proof will be finished if $O' \subseteq \overline{V}'$ will be established, for then we will have $g(z) \in U'$, hence $y \in h(U')$. So choose $b_0' \in O'$. Let W' be a neighborhood of b_0' with $W' \subseteq O'$. As in the proof of theorem 1 we have to establish $V' \cap W' \neq \emptyset$. Choose $b_0 \in E$ with $g(b_0) = b_0'$ and define $W = g^{-1}(W')$, $V = g^{-1}(V')$, $O = g^{-1}(O')$. Repeating the proof of theorem 1, we find $c \in V$, $b \in W$ such that $f(c) = f(b)$. Now this yields $g(c) = g(b)$. But $g(c) \in V'$ and $g(b) \in W'$. Hence the proof is finished. ■

Remark 4. In [Wi] open mapping theorems like our theorem 1 are proved for relations instead of mappings. Since the method of proof is essentially the same in both cases, we did not adopt this more general form here. Essentially relations are used in [BP], too, although the terminology is not the same as in [Wi]. Now, in [Wi], Wilhelm has proved a theorem which, being less general than our theorem 1 as the space E is concerned, yet is more general as the mapping f is concerned. We reproduce his theorem here using functions instead of relations.

Theorem. (Wilhelm [Wi]). *Let E be a Čech-complete space and let f be a nearly open bijection from E to some topological space F. Suppose that for all $x, y \in E$ with $x \neq y$ there exist neighborhoods U, V of x, y respectively such that* $\operatorname{cl}(f(U)) \cap f(V) = \emptyset$. *Then f is open.*

An analysis of Wilhelm's proof shows that E may be assumed to be strictly c-complete and regular. However the proof heavily makes use of the 'strictness'. Therefore it seems as if his theorem did not remain true for c-complete spaces. We will now prove that the assumption of regularity on the space E in Wilhelm's theorem in fact is essential.

Suppose that E is a first countable hausdorff space such that the statement of Wilhelm's theorem is true for all f and F as above. Then E must be a regular space.

Proof. Let τ be the topology on E. Fix $x_0 \in E$ and some decreasing open neighborhood base $(U_n: n \in \mathbb{N})$ of x_0. Now define a new topology τ' on E by taking $\{V \in \tau: x_0 \notin V\} \cup \{\mathrm{cl}(U_n): n \in \mathbb{N}\}$ as a subbase. The identic mapping $1_E: (E, \tau) \to (E, \tau')$ has all the properties mentioned in the statement of Wilhelm's theorem above. This follows from the fact that the closures of the sets U_n are the same with respect to τ and τ'. But then 1_E must be open or, equivalently, $\tau \subseteq \tau'$. But now $(\mathrm{cl}(U_n))$ is a neighborhood base for x_0 in (E, τ) and, consequently, (E, τ) is regular at x_0. ∎

Remark 5. We conclude this section with the remark that the assumption of c-completeness for the space E in theorem 1 is by no means necessary. Indeed theorem 1 will also hold true if E is assumed to be a H-minimal space. However Herrlich has given an example of a H-minimal space which is of the first category ([He]) and consequently cannot be c-complete. Beyond this we note that the space constructed by Herrlich, is first countable. In view of remark 4 this proves that Wilhelm's theorem is not true for H-minimal spaces, even in the weaker form given here.

3. Second open mapping theorem. In this section we prove a variant of our first open mapping theorem. We relax the conditions on both the spaces E, F and consequently, have to assume a little more on the mapping f.

Theorem 3. *Let E be a semi-regular p-complete space. Let f be a continuous, nearly open bijection from E to a topological space F. Suppose that the graph $G(f)$ of f is a set of interior condensation in $E \times F$. Then f is open.*

Proof. Let (ϕ, T) be a web on E with condition (p). Let (ψ, B) be given for $G(f)$ as in 0.3. Our proof now starts exactly as the proof of theorem 1. Let U, V, O, W and x, y, z have the same meaning as in the proof of theorem 1. Again we have to establish $V \cap W \neq \emptyset$.

Still as in theorem 1 we find some $t_1 \in T$ with $\phi(t_1) \subseteq V$ and some $x_1 \in \phi(t_1)$ such that $f(x_1) \in \mathrm{int}(\mathrm{cl}(f(W)))$. But now $(x_1, f(x_1))$ is an element of $G(f)$. By condition (i) in 0.3 there exists $b_1 \in B$ with $(x_1, f(x_1)) \in \psi(b_1)$. Now we select open sets U_1 in E and V_1 in F such that $x_1 \in U_1 \subseteq \phi(t_1)$, $f(U_1) \subseteq V_1 \subseteq \mathrm{int}(\mathrm{cl}(f(W)))$ and $U_1 \times V_1 \subseteq \psi(b_1)$. Next we observe that $\mathrm{int}(\mathrm{cl}(f(U_1))) \cap V_1$ is a neighborhood of $f(x_1) \in \mathrm{cl}(f(W)) = \mathrm{cl}(f(\bigcup\{\phi(s): s \in T,\ \phi(s) \subseteq W\}))$, where the equality follows from condition (i) in 0.2. Hence there exist some $s_1 \in T$ with $\phi(s_1) \subseteq W$ and some $y_1 \in \phi(s_1)$ having $f(y_1) \in \mathrm{int}(\mathrm{cl}(f(U_1))) \cap V_1$. But $(y_1, f(y_1)) \in G(f)$ and therefore there exists $c_1 \in B$ with $(y_1, f(y_1)) \in \psi(c_1)$. Again we choose open sets O_1 in E and W_1 in F such that $y_1 \in O_1 \subseteq \phi(s_1)$, $f(O_1) \in W_1 \subseteq \mathrm{int}(\mathrm{cl}(f(U_1))) \cap V_1$ and $O_1 \times W_1 \subseteq \psi(c_1)$.

Proceeding in this way we define sequences $(x_n), (y_n), (t_n), (s_n), (U_n), (V_n), (O_n), (W_n)$ and $(b_n), (c_n)$ such that the following conditions are satisfied:

(1) $t_{n-1} <_T t_n$, $x_n \in \phi(t_n) \subseteq U_{n-1} \subseteq \phi(t_{n-1})$;

$f(x_n) \in f(U_n) \subseteq V_n \subseteq \mathrm{int}(\mathrm{cl}(f(O_{n-1}))) \cap W_{n-1}$.

(2) $U_n \times V_n \subseteq \psi(b_n)$, where $b_{n-1} <_B b_n$.

(3) $s_{n-1} <_T s_n$, $y_n \in \phi(s_n) \subseteq O_{n-1} \subseteq \phi(s_{n-1})$;
$f(y_n) \in f(O_n) \subseteq W_n \subseteq \text{int}(\text{cl}(f(U_n))) \cap V_n$.

(4) $O_n \times W_n \subseteq \psi(c_n)$, where $c_{n-1} <_B c_n$.

Using property (p) of (ϕ, T) we find $x \in \bigcap\{\phi(t_n): n \in \mathbb{N}\} \subseteq V$, $y \in \bigcap\{\phi(s_n): n \in \mathbb{N}\} \subseteq W$ such that $f(x) \in \bigcap\{V_n: n \in \mathbb{N}\} = \bigcap\{W_n: n \in \mathbb{N}\}$. But then we have $(y, f(x)) \in \phi(s_n) \times W_{n-1} \subseteq O_{n-1} \times W_{n-1} \subseteq \psi(c_{n-1})$ for all $n \geqq 2$, and, by condition (iii) in 0.3, applied to (ψ, B) we have $(y, f(x)) \in G(f)$. But this also reads $f(y) = f(x)$, or $y = x$ since f is injective. Hence we have proved $V \cap W \neq \emptyset$. ■

Corollary 1. *Let E be a semi-regular p-complete space. Let f be a continuous, nearly open bijection from E to a topological space F whose diagonal Δ is a set of interior condensation in $F \times F$. Then f is an open mapping.*

Proof. We only have to prove that the graph $G(f)$ of f is a set of interior condensation. Now for Δ there is given a pair (ϕ, T) as in 0.3. Define ψ by $\psi(t) = \{(x, y) \in E \times F: (f(x), y) \in \phi(t)\}$. $\psi(t)$ is the preimage of $\phi(t)$ under the continuous mapping $(x, y) \to (f(x), y)$ and consequently is open. Now the graph $G(f)$ of f is a set of interior condensation with (ψ, T). ■

Remark 5. Theorem 1 cannot be derived from theorem 3, because the graph of a continuous mapping f need not be a set of interior condensation.

The question, whether the bijectivity of f can be replaced by injectivity may be answered along the route of remark 2. Using the example from [BP], it can be shown that bijectivity cannot be relaxed to surjectivity, here. However, a statement like theorem 2 might be derived from corollary 1 above. We leave the details to the reader.

4. Semitopological groups. In this section we apply our two open mapping theorems in semitopological and topological groups. Following Husain ([Hu]), a topological space G is called a *semitopological group* if G is a group and if the right and left translates $x \to xa$, $x \to ax$ are all continuous with respect to the topology on G. Of course every topological group is a semitopological group.

Theorem 4. *Let G be a semitopological group. Suppose that H is a dense subgroup of G which is semi-regular in its relative topology. Suppose further that one of the following statements* (a) *or* (b) *is satisfied. Then $H = G$.*

(a) *H is c-complete and G is hausdorff.*

(b) *H is p-complete and the diagonal in G is a set of interior condensation.*

Proof. Assume that there exists $z \in G \setminus H$. Let $F := zH \cup H$ be endowed with the trace of the topology of G. Now let E be the topological sum of two copies of H. We may assume that $E = H \times \{1\} \cup H \times \{2\}$. Now define a mapping $f: E \to F$ by $f(x, 1) = zx$, $f(x, 2) = x$ for all $x \in H$. Thus f is bijective since $zH \cap H = \emptyset$. Moreover, f is continuous and nearly open, the latter since zH and H are both dense in F. For the first assume that (a) holds true. Then F is hausdorff as a subspace of G. H being semi-regular and c-complete, the same must be true for E. But then f is open by theorem 1. Next assume that (b) holds true. Then the diagonal in G is a set of interior condensation and, consequently, the same is true in F. But now, E is p-complete and semi-regular since H is. Therefore by corollary 1, f is open. But now we see that both cases yield a contradiction, for $H \times \{1\}$ being open in E, the same must be true for zH in F, contradicting the denseness of H in F. ■

Corollary 2. *Let E be a semi-regular semitopological group. Let f be a continuous, nearly open homomorphism from E to some semitopological group F. Suppose that f is injective and that $f(E)$ is dense in F. Then f is open and surjective, provided that one of the following conditions is satisfied*:

(a) *E is c-complete and F is hausdorff.*

(b) *E is p-complete and the diagonal in F is a set of interior condensation.*

Proof. In both cases, f is open as a mapping $E \to f(E)$ by theorem 1 respectively corollary 1. But now apply theorem 4 to the dense subgroup $f(E)$ of F. ■

Remark 6. We have proved that for semitopological groups E, F and a homomorphism f the bijectivity in the open mapping theorem may be replaced by dense injectivity. We note that if E is a topological group, then injectivity may be omitted, too. Indeed, we may then take the quotient group $E/\mathrm{Ker}\, f$ to regain injectivity. Note that, the canonical mapping being open, $E/\mathrm{Ker}\, f$ is again c-resp. p-complete, hence we can apply theorem 1 resp. corollary 1. The reason why we cannot do the same in case that E is semitopological is: when taking the quotient group $E/\mathrm{Ker}\, f$ we do not know whether this space is semi-regular.

5. Barrelledness. In the classical theory of open mapping theorems, carried out in the category of locally convex vector spaces, the class of barrelled spaces plays an important role in view of the following statement ([Kö], p. 24):

Every surjective linear mapping f from an arbitrary locally convex space E onto a barrelled locally convex space F is nearly open.

In the category of topological groups there does not exist a class of groups which might play a comparably important role, for a group F which yields the analogon of the statement above to hold true for all E, f, obviously must be discrete. However, in certain more special situations, the class of Baire topological groups can play the role of barrelledness:

Suppose that E is a separable or Lindelöff topological group (more generally a σ-bounded group (see [Pe])*). Then every surjective homomorphism f from E onto a second category topological group F is nearly open* (See [Hu], p. 98, [Pe]).

We will now give a definition of 'barrelledness' in the categories of semitopological and topological groups which takes into consideration the proposition above. First however, we need some aids.

Definition 3. Let E be a semitopological group. Let V be a neighborhood of the unit e in E. A set $\mathscr{R}$ of open subsets of E is called a *V-base* if for every $x \in E$ there exists $B \in \mathscr{R}$ such that $x \in B \subseteq xV$. ∎

Let us introduce the following notion. A mapping f from a group G to a group H is called *left-invariant* if for every $a \in G$ there exists $b \in H$ such that for all $x \in G$ we have $f(ax) = b\,f(x)$. Every multiplicative mapping is left-invariant.

Definition 4. Let E, F be semitopological groups. Let f be a mapping from E to F. f is called a *σ-mapping* if for every neighborhood V of e in F there exists some V-base $\mathscr{R}$ such, that $f^{-1}(\mathscr{R})$ is σ-d.d. in E. (See 0.4). If, in addition, f is left-invariant resp. a homomorphism, then f is called an *invariant σ-mapping* resp. a *σ-homomorphism*. ∎

Examples. 1) Let E, F be metrizable semitopological groups. Then a mapping $f: E \to F$ is σ-discrete in the sense of Hansell if and only if it is a σ-mapping in the sense of definition 4. (Compare [Ha] for examples of σ-discrete mappings).

2) Let F be a σ-bounded topological group ([Pe]). Then every mapping f from an arbitrary semitopological group E to F is a σ-mapping.

Proof. Let V be a neighborhood of e in F. Choose some symmetric neighborhood U of e with $UU \subseteq V$. By the definition of σ-boundedness there exists a sequence (x_n) of elements of F with $F = \bigcup\{x_nU: n \in \mathbb{N}\} \cup \bigcup\{Ux_n: n \in \mathbb{N}\}$. But now the set $\mathscr{R} = \{x_nU: n \in \mathbb{N}\} \cup \{Ux_n: n \in \mathbb{N}\}$ is a countable V-base and, consequently, $f^{-1}(\mathscr{R})$ is σ-d.d. ∎

This statement is no longer true if F is a separable or Lindelöff semitopological group. Indeed, let S be the Sorgenfrey space ([E], p. 39). S is a separable and Lindelöff semitopological group. However the homomorphism $1_{\mathbb{R}}: \mathbb{R} \to S$ is not a σ-homomorphism as we will see below.

3) Note that in Hansell's theory every continuous mapping f between metrizable spaces is σ-discrete (cf. [Ha]). In our situation this is not true a fortiori. However if F is assumed to have σ-discrete V-bases for each of its e-neighborhoods V, then every continuous $f: E \to F$, with E an arbitrary semitopological group, is a σ-mapping. In particular, this is the case if F is metrizable or if it is a topological group.

Definition 5. A semitopological group (resp. a topological group) E is called *of type b* (resp. *of type b in the category of topological groups*) if every bijective, left-invariant σ-mapping (resp. every bijective σ-homomorphism) f from E to an arbitrary semitopological (resp. topological) group F is nearly continuous. ∎

Remark 7. The name "b-type" has been chosen in reminiscence of the name "barrelled". Note that b-type topological groups may be defined by the following formally weaker condition: a topological group E is of b-type in the category of topological groups if and only if every surjective σ-homomorphism $f: E \to F$, F being any topological group, is nearly continuous.

The following theorem justifies definition 5.

Theorem 5. *Every second category semitopological group is of b-type. Every second category topological group is of b-type in the category of topological groups.*

Proof. Of course we only have to prove the first part of the statement. Let $f: E \to F$ be surjective, left-invariant and a σ-mapping. Suppose that E is of the second category. Assume that f is not nearly continuous. Hence there exist $x \in E$ and an open neighborhood U of $f(x)$ such that $x \notin \operatorname{int}(\operatorname{cl}(f^{-1}(U)))$. Now $f(x)^{-1} U$ is a neighborhood of e. Since f, by assumption, is a σ-mapping, there exists a $f(x)^{-1} U$-base $\mathscr{R}$ such that $f^{-1}(\mathscr{R})$ is σ-d.d., $f^{-1}(\mathscr{R}) = \bigcup\{\mathscr{R}_n: n \in \mathbb{N}\}$ with $\mathscr{R}_n$ discrete in E. Now $X := \bigcup\{f^{-1}(yU) \setminus \operatorname{int}(\operatorname{cl}(f^{-1}(yU))): y \in F\} \subseteq \bigcup\{f^{-1}(R) \setminus \operatorname{int}(\operatorname{cl}(f^{-1}(R))): R \in \mathscr{R}\}$ since $\mathscr{R}$ is a $f(x)^{-1} U$-base. By 0.5 we may continue $X \subseteq \bigcup\{f^{-1}(R) \setminus \operatorname{int}(\operatorname{cl}(f^{-1}(R))): R \in \mathscr{R}\} \subseteq \bigcup\{M \setminus \operatorname{int} D(M): M \in f^{-1}(\mathscr{R})\} \subseteq \bigcup\{\bigcup\{M \setminus \operatorname{int} D(M): M \in \mathscr{R}_n\}: n \in \mathbb{N}\}$. By 0.5 each $M \setminus \operatorname{int} D(M)$ is of the first category. But the $\mathscr{R}_n$ being discrete, the sets $\{M \setminus \operatorname{int} D(M): M \in \mathscr{R}_n\}$ are also discrete and, by the Banach-category theorem, their union is of the first category. Therefore we have proved that X is of the first category and, E being of the second category, there exists $z \in E \setminus X$. Now by the left-invariance of f we find $b \in F$ such that for all $y \in E$ we have $f(zx^{-1}y) = b f(y)$. Hence $f(z) \in bU$ and by the definition of X now $z \in \operatorname{int}(\operatorname{cl}(f^{-1}(bU)))$. However this yields $z \in zx^{-1} \operatorname{int}(\operatorname{cl}(f^{-1}(U)))$, a contradiction. ∎

Remark 8. We can now prove that $1_{\mathbb{R}}: \mathbb{R} \to S$ is not a σ-homomorphism. Indeed, suppose it were. Then by theorem 5 it would be nearly continuous, respectively its inverse $1_{\mathbb{R}}: S \to \mathbb{R}$ would be nearly open. But the Sorgenfrey topology is finer than the euclidean topology and moreover is regular and p-complete. In view of theorem 3 this would mean that $1_{\mathbb{R}}: S \to \mathbb{R}$ had to be open. This however is not the case.

In the category of topological groups we can prove a more general result than theorem 5. Namely we can prove that there are b-type topological groups which are of the first category. Such a group may be constructed in the following way. Let G be a group and let $(G_n: n \in \mathbb{N})$ be an increasing sequence of subgroups of G with union G. Suppose that on each G_n there is given a group topology τ_n such that the trace of τ_{n+1} in G_n is τ_n. Assume that there exists a finest group topology τ on G which induces τ_n on G_n for every n. In this case (G, τ) is called the *inductive limit of the sequence* $((G_n, \tau_n): n \in \mathbb{N})$. Note that if G is commutative, then τ always exists.

Theorem 6. *Let (G, τ) be the inductive limit of a sequence $((G_n, \tau_n): n \in \mathbb{N})$ of b-type topological groups. Then G is itself a b-type topological group. In particular there exist b-type topological groups which are of the first category.*

Proof. Let $f: G \to H$ a bijective σ-homomorphism onto a topological group H. Let f_n be the restriction $f \mid G_n$. Let $\mathscr{L}$ be the set of all neighborhoods of e in F. Let $\mathscr{L}^*$ be the set of $\operatorname{cl}(f^{-1}(N))$, $N \in \mathscr{L}$. $\mathscr{L}^*$ is the neighborhood base for a new group topology σ on G. But the trace σ_n of σ on G_n is coarser than τ_n. In fact, we have the inclusion $\operatorname{cl}_n(f^{-1}(U) \cap G_n) \subseteq \operatorname{cl}(f^{-1}(U)) \cap G_n$, where cl_n is the closure operator in G_n. However, f_n being a σ-homomorphism, too, each f_n is nearly continuous. Hence $\operatorname{cl}_n(f_n^{-1}(U)) = \operatorname{cl}_n(f^{-1}(U) \cap G_n)$ is a neighborhood of e in G_n and so is $\operatorname{cl}(f^{-1}(U)) \cap$ $\cap\, G_n$. But now, $\sigma_n \subseteq \tau_n$ for each n, yields $\sigma \subseteq \tau$ by the definition of τ.

For the second part of the statement we only have to choose the sequence (G_n)

such that the limit topology exists and $\operatorname{int}(\operatorname{cl}(G_n)) = \emptyset$ for all n. Then G will be of the first category. ■

The remainder of this section is devoted to an application of the concept of b-type topological groups. Hereby we also apply an open mapping theorem. First we need a definition.

Let F be a metrizable topological space, d a metric for F. Let $c(F)$ be the set of all convergent sequences of elements of F. Define a metric $\tilde{d}$ on $c(F)$ by $\tilde{d}(x, y) = \sup\{d(x_n, y_n): n \in \mathbb{N}\}$. Of course the topology induced by $\tilde{d}$ does not depend on the choice of d. Note that the new topology is finer than the product topology on $c(F)$. Now we have the following

Lemma 1. *There exists a network* (cf. [E], p. 170) $\mathscr{R}$ *on* $c(F)$, *which is* σ-*discrete with respect to the coarser product topology on* $c(F)$. *Moreover* $\mathscr{R}$ *consists of sets which are* F_σ *with respect to the product topology.*

Proof. Let $\mathscr{B}$ be a σ-discrete base for F ([E], p. 350). Let $\mathscr{R}_n$ be the set of all sets $R(B_1, \dots, B_n) = \Pi\{B_i: i \in \mathbb{N}\} \cap c(F)$, where $B_i \in \mathscr{B}$ and $B_i = B_n$ for $i > n$. $\mathscr{R}_n$ is σ-discrete with respect to the trace of the product topology on $c(F)$. But now $\mathscr{R} = \bigcup\{\mathscr{R}_n: n \in \mathbb{N}\}$ is a network on $c(F)$. Indeed, let $x \in c(F)$ and $\varepsilon > 0$ be fixed. Choose $B \in \mathscr{B}$ with d-diameter less than $\varepsilon/3$ and $x_i \in B$ for all $i \geqq n$. For $i > n$ choose B_i such that $x_i \in B_i$ and the diameters of the B_i are less than ε. Let now $B_i = B$ for all $i \geqq n$, then $x \in R(B_1, \dots, B_n) \subseteq B_{\tilde{d}}(x, \varepsilon)$. The second part of the statement is obvious. ■

Suppose now that $(f_n: n \in \mathbb{N})$ is a sequence of continuous mappings from a topological space E to the metrizable space F. Then $\phi(x) := (f_n(x): n \in \mathbb{N})$ defines a mapping $\phi: E \to F^{\mathbb{N}}$, which is continuous with respect to the product topology on $F^{\mathbb{N}}$. Suppose now that E, F are topological groups and that the $f_n: E \to F$ are homomorphisms. Suppose further that for every $x \in E$ the sequence $(f_n(x))$ is convergent. Then ϕ maps E into $c(F)$. In this case we have:

Lemma 2. ϕ *is a* σ-*homomorphism. Moreover,* ϕ *is Borel-measurable of the class one* (*i.e. preimages of open sets are* F_σ-*sets*).

Proof. ϕ is continuous with respect to the product topology. But $\mathscr{R}$, defined as in lemma 2, is σ-discrete and consists of F_σ-sets, both with respect to the product topology. Hence the same is true for $\phi^{-1}(\mathscr{R})$. Since $\mathscr{R}$ is a network, this proves the lemma. ■

Now we can state and prove the following variant fo the Banach Steinhaus theorem in the category of topological groups.

Theorem 7. *Let* E, F *be commutative, separated topological groups. Let* E *be of* b-*type. Let* (f_n) *be a sequence of continuous homomorphisms from* E *to* F *such that for every* $x \in E$ *the sequence* $(f_n(x))$ *is convergent. Then the set* $\{f_n: n \in \mathbb{N}\}$ *is equicontinuous, the limit homomorphism* f *is continuous and* (f_n) *converges to* f *uniformly on every precompact subset of* E.

Proof. Since F is commutative and separated, it can be embedded into a product of completely metrizable commutative groups. Therefore it will be sufficient to prove the statement for commutative, completely metrizable F. But then $c(F)$ and ϕ as above, are available. Let τ be the topology on $c(F)$. $\phi: E \to (c(F), \tau)$ is a σ-homomorphism by lemma 2 and, E being of b-type, is nearly continuous by the remark following definition 5. We define a new group topology σ on $c(F)$ by taking for a base of the filter of neighborhoods of e of σ the sets of the form $\phi(V) . U$, where V runs through the neighborhoods of e in E and U runs through the neighborhoods of e in $(c(F), \tau)$. But now $\phi: E \to (c(F), \sigma)$ is continuous and on the other hand $1_{c(F)}$: $(c(F), \tau) \to (c(F), \sigma)$ is bijective, continuous and nearly open, the latter since ϕ is nearly continuous. But $c(F)$ is completely metrizable, for d being a complete metric for F, the metric $\tilde{d}$ on $c(F)$ is complete, too. Since ϕ is continuous with respect to the product totpology, it has closed graph with respect to τ. This however guarantees that σ is a hausdorff topology. Consequently, by theorem 1, $1_{c(F)}$ is open, which also reads $\tau = \sigma$. But then $\phi: E \to (c(F), \tau)$ is continuous. This however means nothing else but the equicontinuity of the sequence (f_n). The rest of the statement now follows by a well-known argument (see for example [Bo], X, § 2.4, th. 1). ■

Remark 9. Compare theorem 6 with corollary 2.1 in [Pe].

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