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TOEPLITZ SECTIONS AND THE WILANSKY PROPERTY

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Abstract. We discuss the general problem of when a perfect permanent summability method A is uniquely determined among all other perfect permanent summability methods by means of an appropriate space $(c_A \rightarrow c_T)$ of summability factor sequences. As applications we obtain that the Cesàro convergence domains c_{C_α}, $\alpha > 1$, are uniquely determined among the convergence domains of all perfect permanent methods by their $C_{\alpha-1}$-summability factor sequences, while the Cesàro summability domains $c_{C_\alpha S}$, $\alpha > 0$, are uniquely determined among the summability domains of all perfect permanent methods by their C_α-summability factor sequences.

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Introduction. Let A be a matrix summability method with convergence domain c_A. Let c_A^β denote the corresponding set of summability factor sequences. Not only is it of theoretical interest, but also of practical relevance (cf. [7,10]) to ask whether c_A is uniquely determined by its set of summability factors c_A^β. More precisely, given any other matrix summability method B having the same set of summability factors $c_B^\beta = c_A^\beta$, is it true that A and B are equivalent, i.e. $c_A = c_B$? The answer is in the negative, in general, as may be seen from the following example. Let a,b be the sequences defined by $a_n = n$, $b_n = n^2$. Using Mazur's construction (cf. [15, S. 48]), we find matrices A,B having $c_A = c + lin\{a,b\}$, $c_B = c + lin\{b\}$. Clearly $c_B \neq c_A$, but

$$c_A^\beta = c_B^\beta = \{x \in \omega: \sum n^2 x_n \; converges\}.$$

The problem being ill-posed in the general context, there is more hope when we restrict our considerations to perfect methods. For instance, it follows from a general result of Snyder and Wilansky (cf. [14, 7.2.7]) that if A is a matrix method having sectional convergence (i.e. $(c_o)_A$ is an AK-space), then c_A is uniquely determined by its set of summability factors c_A^β among all perfect methods. Unfortunately, sectional convergence is quite a restrictive requirement to impose on the null-domain of a matrix method. In [6] we therefore proved a stronger result, stating that for a perfect matrix method A whose space of summability factors c_A^β is separable in its intrinsic BK-topology, the coincidence of the summability factors $c_A^\beta = c_B^\beta$ implies $c_A = c_B$ for any (not necessarily perfect) method B weaker than A. In other terms, c_A is *minimal* with respect to its set of summability factors c_A^β.

It is well-known that a BK-convergence domain c_A has separable strong dual c_A'. So its space of summability factors c_A^β is always separable when considered part of the topological dual. But surprisingly enough, the assumption that c_A^β be separable in its own BK-topology is a stronger requirement. For instance, consider the Cesàro method C_2 of the second order. This provides an example of a convergence domain with non-separable factor space $c_{C_2}^\beta$. So even in this case our results from [6] do not guarantee that $c_{C_2}^\beta$ is determined by its summability factors among all perfect methods.

The example of the Cesàro methods C_α of order $\alpha > 1$ suggests considering other kinds of factor sequences. For instance, we shall prove here that c_{C_α} may be reconstructed from its set of $C_{\alpha-1}$-summability factors $(\alpha > 1)$. In other terms, replacing the space of summability factors $c_{C_\alpha}^\beta = (c_{C_\alpha} \to c_S)$ by the factor space $(c_{C_\alpha} \to c_{C_{\alpha-1}S})$ permits identifying the Cesàro method C_α among all perfect summability methods. The reason for this is easily found. While C_α does not have sectional convergence, it has $C_{\alpha-1}$-summable sections (in the sense of [15]), so $(c_o)_{C_\alpha}'$, the dual of $(c_o)_{C_\alpha}$, may be identified with $(c_{C_\alpha} \to c_{C_{\alpha-1}S})$, hence this factor space is again separable in its own BK-topology. The purpose of the present paper is to study these aspects of factor sequence spaces in detail. Replacing the Cesàro summation matrices $C_\alpha S$ by a general Toeplitz lower triangular matrix T with column limits 1, we ask for conditions under which a perfect c_A is uniquely determined by some factor space of the type $(c_A \to c_T)$ among all perfect summability methods. In particular, we obtain applications to the case of Cesàro methods C_α mentioned above.

Preliminaries. Generally our terminology is based on the books [14, 15]. In the following we briefly discuss some additional notions needed here, in particular the concept of Toeplitz sections and Toeplitz sectional convergence. More details concerning the latter notions may be found in [2, 4].

Let E,F be sequence spaces. We denote by $(E \to F)$ the space of all sequences $x \in \omega$ having $x \cdot y \in F$ for all $y \in E$. Here $x \cdot y$ denotes the coordinatewise product of the sequences x,y. In particular we have $E^\beta = (E \to c_S)$ and $E^\gamma = (E \to (\ell_\infty)_S)$.

Throughout let T be a lower triangular matrix whose columns have limit 1. Given any sequence $x \in \omega$, we denote by

$$t^n \cdot x = (t_{n1}x_1, \dots, t_{nn}x_n, 0, 0, \dots),$$

$n \in \mathbb{N}$, the T-sections of x. Here t^n denotes the nth row of T. In the case where T is the summation matrix S we obtain the usual sections $s^n x = (x_1, \dots, x_n, 0, 0, \dots)$ of the sequence x. An FK-space E is said to have T-summable sections or T-sectional convergence (TAK for short) if it contains Φ and, for every $x \in E$, the T-sections $t^n x$ of x converge to x in the sense of the topology of E.

Let E be a sequence space. We denote by E^{β_T} (resp. E^{γ_T}) the space of all sequences $y \in \omega$ such that $\sum_{i=1}^{n} t_{ni}x_i y_i$ converges (resp. is bounded) for every $x \in E$. Observe that $E^{\beta_T} = (E \to c_T)$, $E^{\gamma_T} = (E \to (\ell_\infty)_T)$ in our terminology used above. In the case where T is the summation matrix S, we shall write again β,γ instead of β_S,γ_S.

Let E be an FK-space with TAK. Then E', its topological dual, may be identified with E^{β_T} using the bilinear form

$$\langle x,y \rangle = \lim_{n\to\infty} \sum_{i=1}^{n} t_{ni}x_i y_i.$$

In the case where E is a BK-space, E^{β_T} and E^{γ_T} are as well BK-spaces with the norm $\| \, \|_T$ defined by

$$\|y\|_T = \sup_{\|x\|\le 1, x\in E} \; \sup_{n\in\mathbb{N}} \Big| \sum_{i=1}^{n} t_{ni}x_i y_i \Big|. \tag{1}$$

Similarly, starting with the norm $\| \|_T$ on E^{β_T} and using (1), we obtain the norm $\| \|_{TT}$ on $E^{\beta_T \beta_T}$. Notice that $\| \|_T$ also has the representation

(2) $$\|y\|_T = \sup_{n \in \mathbb{N}} \|t^n y\|,$$

where on the right hand side $\| \|$ denotes the dual norm on E', the elements of Φ being identified with elements of E'. Indeed, (2) may be obtained from (1) by interchanging the sups.

The Main Theorem. This section presents our first central result, which is technical in nature. Antecedents may be found in [1,6,8,13].

Theorem 1. *Let E be a BK-space containing* Φ. *Let T be a lower triangular matrix with column limits 1, and suppose* E^{γ_T} *is separable with* $\| \|_T$. *Let F be a dense subspace of E containing* Φ. *Then the following statements are equivalent:*

(1) $F^{\beta_T} = E^{\beta_T}$;

(2) $F^{\gamma_T} = E^{\gamma_T}$;

(3) $F^{\beta_T} \subset E^{\gamma_T}$;

(4) Φ *has the same null sequences with respect to the weak topologies* $\sigma(\Phi,E)$ *and* $\sigma(\Phi,F)$;

(5) Whenever A is a lower triangular matrix, then $F \subset c_A$ *implies* $E \subset c_A$.

Proof. (1) implies (2). Indeed, $F^{\beta_T} = E^{\beta_T}$ gives $F^{\beta_T \gamma_T \gamma_T} = E^{\beta_T \gamma_T \gamma_T}$. Since the equality $H^{\beta_T \gamma_T \gamma_T} = H^{\gamma_T}$ holds for every H (see [2] and [3]), we obtain (2). The fact that (2) implies (3) is obvious. The hard implication is (3) to (4). (4) implies (5): Observe that the coincidence of null sequences implies the coincidence of Cauchy sequences. Now $F \subset c_A$ means that the sequence (a^n) of rows of A is Cauchy in $\sigma(\Phi,F)$, hence is Cauchy with respect to $\sigma(\Phi,E)$, giving $E \subset c_A$. (5) implies (1). Since the inclusion $E^{\beta_T} \subset F^{\beta_T}$ is clear, we check the reverse inclusion. Let $a \in F^{\beta_T}$ and choose for A the matrix whose nth row is $t^n a$. Then we have $F \subset c_A$, giving $E \subset c_A$, which is just the desired statement $a \in E^{\beta_T}$. So proving that (3) implies (4) remains. This requires six steps.

I. Clearly $F \subset E$ implies that $\sigma(\Phi,E)$-null sequences are as well $\sigma(\Phi,F)$-null. So we have to prove the reverse implication. Observe that it suffices to show that every $\sigma(\Phi,F)$-null sequence (y^n) is bounded in the dual norm $\| \ \|$. For suppose this has been established, $\|y^n\| \leq M$, $n \in \mathbb{N}$, say. Fixing $x \in E$ and $\varepsilon > o$, we find $\tilde{x} \in F$ having $\|x - \tilde{x}\| < \varepsilon/2M$. This gives

$$\begin{aligned} |\langle x,y^n\rangle| &\leq |\langle x-\tilde{x},y^n\rangle| + |\langle \tilde{x},y^n\rangle| \\ &\leq \|x-\tilde{x}\|\cdot\|y^n\| + |\langle \tilde{x},y^n\rangle| \\ &\leq \varepsilon/2 + \varepsilon/2 , \end{aligned}$$

for n large enough. So (y^n) is $\sigma(\Phi,E)$-null.

II. Suppose the $\sigma(\Phi,F)$-null sequence (y^n) is not bounded with respect to the dual norm, $\|y^n\| \geq 2^n$, say. Since $\Phi \subset F$, the sequence (y^n) must be coordinatewise null. We therefore find strictly increasing sequences (k_j), (n_j) of indices satisfying

(a) y^{n_j} has length at most k_j,

(b) $\|y^{n_j} - s^{k_{j-1}}\cdot y^{n_j}\| \geq 2^j$,

$j = 1,2,\ldots.$ Here $s^r x$ denotes the usual rth section $(x_1, \ldots, x_r, o, o, \ldots)$ of x.

Indeed, suppose $k_1, \ldots, k_j$ and $n_1, \ldots, n_j$ have been defined according to (a) and (b). Since $\|s^{k_j}\cdot y^n\| \to o \ (n \to \infty)$, we find $n_{j+1} > n_j$ satisfying (b). Since $y^{n_{j+1}} \in \Phi$ has finite length $\leq k_{j+1}$, statement (a) is as well satisfied.

Let $u^j = y^{n_j} - s^{k_{j-1}}\cdot y^{n_j}$, $j = 1,2,\ldots.$ Then we have $\|u^j\| \geq 2^j$ and at the same time $u^j \to o \ (j \to \infty)$ with respect to $\sigma(\Phi,F)$.

III. We select a subsequence from the sequence (u^j). We claim the existence of strictly increasing sequences (r_j), (m_j) of indices satisfying

(c) $t^{r_j}\cdot u^{m_{j+1}} = o, \quad j = 1,2,\ldots,$

(d) $\|t^r\cdot \sum_{i=1}^{m_j} \lambda_i u^i - \sum_{i=1}^{m_j} \lambda_i u^i\| \leq 2^{-j-2}$ for all $r \geq r_j$ and all λ_i, $1 \leq i \leq m_j$ having $|\lambda_i| \leq 1, \quad j = 1,2,\ldots.$

Suppose $r_1, \ldots, r_j$ and $m_1, \ldots, m_j$ have been defined in accordance with (c) and (d). By the definition of the vectors u^i we first find an index $m_{j+1} > m_j$ such that $t^{r_j} \cdot u^{m_{j+1}} = o$. Now we observe that

$$\lim_{r\to\infty} \sup_{\substack{|\lambda_i| \le 1 \\ 1\le i\le m_j}} \| t^r \cdot \sum_{i=1}^{m_{j+1}} \lambda_i u^i - \sum_{i=1}^{m_{j+1}} \lambda_i u^i \| = o ,$$

since T has column limits 1. The uniformity of the limit over the region $|\lambda_i| \le 1$, $i = 1, \ldots, m_j$ may be proved using a compactness argument. But this provides an index $r_{j+1} > r_j$ in accordance with condition (d).

IV. For each j we choose an index $r(j) \in \{r_{j-1}+1, \ldots, r_j\}$ such that

$$\|t^{r(j)} \cdot u^{m_j}\| = \max \{ \|t^r \cdot u^{m_j}\| : r_{j-1} < r \le r_j \}.$$

Now let $\alpha_j = 1/\|t^{r(j)} \cdot u^{m_j}\|$, $z^j = \alpha_j u^{m_j}$, $j = 1,2,\ldots$. Observe that by condition *(d)*, (α_j) is an ℓ_1-sequence. Let $c_o(z^i)$ be the space of all sequences of the form

$$z = \sum_{i=1}^{\infty} \lambda_i z^i , \quad (\lambda_i) \in c_o ,$$

where summation is understood in the pointwise sense. Analogously we define the space $\ell_\infty(z^i)$. We claim that $\ell_\infty(z^i) \subset c_o(z^i)^{\gamma_T \gamma_T}$.

First observe that $x \in c_o(z^i)^{\gamma_T}$ is true if and only if

(i) $(\langle x, z^j \rangle) \in \ell ,$

(ii) $\langle x, t^r \cdot z^j \rangle = O(1) , \quad (r \to \infty, r_{j-1} < r \le r_j)$

are satisfied. To see this we consider the equality $(r_{j-1} < r \le r_j)$

$$(*) \quad \sum_{i=1}^{r} t_{ri} x_i z_i = \sum_{i=1}^{j-1} \sum_{s=r_{i-1}+1}^{r_i} t_{rs} \lambda_i \alpha_i x_s u_s^{m_i} + \sum_{s=r_{j-1}+1}^{r} t_{rs} \lambda_j \alpha_j x_s u_s^{m_j} .$$

Boundedness of the left hand side for arbitrary $z \in c_o(z^i)$ yields condition *(i)* when we first insert indices $r = r_j$. Here we use the fact that

$$\sum_{s=r_{i-1}+1}^{r_i} t_{rs} x_s u_s^{m_i} = \langle x, t^r \cdot u^{m_i} \rangle = \langle x, u^{m_i} \rangle + \langle x, t^r \cdot u^{m_i} - u^{m_i} \rangle$$

holds with the estimate

$$|\langle x, t^r \cdot u^{m_i} - u^{m_i} \rangle| \leq \|x\|_{TT} 2^{-i-2} \qquad \textit{(using (d))}.$$

Condition *(ii)* then results when we evaluate the second term on the right hand side of (*) for an arbitrary choice of $(\lambda_i) \in c_o$.

Having proved the description of the vectors $x \in c_o(z^i)^{\gamma_T}$ in terms of the conditions *(i)*, *(ii)*, it readily follows that the right hand side of (*) also remains bounded when inserting on the left side arbitrary vectors $z \in \ell_\infty(z^i)$. This proves the claimed inclusion $\ell_\infty(z^i) \subset c_o(z^i)^{\gamma_T \gamma_T}$.

V. We next prove the existence of a null sequence (λ_i) such that $z = \sum \lambda_i z^i$ is *not* an element of E^{γ_T}. Indeed, if we had $c_o(z^i) \subset E^{\gamma_T}$, then by the above $\ell_\infty(z^i) \subset c_o(z^i)^{\gamma_T \gamma_T} \subset E^{\gamma_T \gamma_T \gamma_T} = E^{\gamma_T}$. This, however, is impossible, as we shall see now. In fact, we show that $\ell_\infty(z^i) \cong \ell_\infty$ and $\ell_\infty(z^i)$ is closed in E^{γ_T}. This contradicts the separability of E^{γ_T}.

We define a linear operator ϕ: $\ell_\infty(z^i) \to \ell_\infty$ by setting

$$\phi(z) = \phi(\sum \lambda_i z^i) = (\lambda_i), \quad z \in \ell_\infty(z^i).$$

Clearly ϕ is a linear bijection. We prove that it is continuous. Let $z = \sum \lambda_i z^i$ be fixed. We have

$$\begin{aligned} |\lambda_j| &= \|\lambda_j \alpha_j t^{r(j)} u^{m_j}\| = \|t^{r(j)} \sum_{i=1}^{j} \lambda_i \alpha_i u^{m_i} - t^{r(j)} \sum_{i=1}^{j-1} \lambda_i \alpha_i u^{m_i}\| \\ &\leq \|t^{r(j)} \sum_{i=1}^{j} \lambda_i \alpha_i u^{m_i}\| + \|t^{r_{j-1}} \sum_{i=1}^{j-1} \lambda_i \alpha_i u^{m_i}\| + \\ &\quad \|(t^{r(j)} - t^{r_{j-1}}) \sum_{i=1}^{j-1} \lambda_i \alpha_i u^{m_i}\| \\ &\leq 2 \cdot \sup_r \|t^r z\| + \sup_i |\lambda_i \alpha_i| \cdot 2 \cdot 2^{-j-1} \quad \textit{(using (c),(c) and (d))} \end{aligned}$$

$$\leq 2 \cdot \|z\|_T + 1/2 \cdot \|\lambda\|_\infty .$$

We thus obtain the estimate

$$\|\lambda\|_\infty = \|\phi(z)\|_\infty \leq 4 \cdot \|z\|_T ,$$

proving the continuity of ϕ.

Let H denote the closure of $\ell_\infty(z^i)$ in E^{γ_T}. Then ϕ extends to a continuous linear mapping $\bar\phi$: $H \to \ell_\infty$. As E^{γ_T} is a BK-space, the $z \in H$ are still of the form $z = \sum\lambda_i z^i$, so $\bar\phi(z) = (\lambda_i) \in \ell_\infty$. Hence by the definition of $\ell_\infty(z^i)$ we have $z \in \ell_\infty(z^i)$, proving that $\ell_\infty(z^i)$ is closed in E^{γ_T}. Now the open mapping theorem proves that ϕ is an isomorphism, i.e. $\ell_\infty(z^i) \cong \ell_\infty$. So finally E^{γ_T} has a subspace isomorphic with ℓ_∞, the desired contradiction.

VI. Let (λ_i) be a null sequence chosen in accordance with V., i.e. $z = \sum\lambda_i z^i \notin E^{\gamma_T}$. We end our proof by showing that $z \in F^{\beta_T}$, which contradicts our assumption (3).

Fix $x \in F$ and $r \in \mathbb{N}$, $r_{j-1} < r \leq r_j$. Then we have

$$\sum_{i=1}^{r} t_{ri} x_i z_i = \sum_{i=1}^{j-1} \sum_{s=r_{i-1}+1}^{r_i} t_{rs}\lambda_i\alpha_i x_s u_s^{m_i} + \sum_{s=r_{j-1}+1}^{r} t_{rs}\lambda_j\alpha_j x_s u_s^{m_j}.$$

Here the first term on the right hand side converges $(r \to \infty,\ r_{j-1} < r \leq r_j)$ in view of $(\lambda_j\alpha_j) \in \ell$ and the estimate

$$\Big|\sum_{s=r_{i-1}+1}^{r_i} t_{rs}x_s u_s^{m_i}\Big| = |\langle x, t^r u^{m_i}\rangle| \leq |\langle x, u^{m_i}\rangle| + |\langle x, t^r u^{m_i} - u^{m_i}\rangle|$$

$$\leq |\langle x, u^{m_i}\rangle| + \|x\| \cdot 2^{-i-2} = o(1).$$

The second term on the right hand side converges as well in view of $\lambda_j \to o$ $(r \to \infty,\ r_{j-1} < r \leq r_j)$ and because of the estimate

$$\Big|\sum_{s=r_{j-1}+1}^{r} t_{rs}x_s\alpha_j u_s^{m_j}\Big| = |\langle x, t^r z^j\rangle| \leq \|x\| \cdot \|t^r z^j\| \leq \|x\|,$$

which is based on the definition of the α_j. This yields $z \in F^{\beta_T}$ and hence ends our proof. □

Wilansky property. In this section we shall be concerned with the problem presented in the exposition - first on a more abstract level. We start by recalling a definition from [1]. An *FK*-space E containing Φ is said to have the *Wilansky property (W)* if every dense *FK*-subspace F of E satisfying $F^{\beta} = E^{\beta}$ must coincide with E, i.e. $F = E$.

This property was first considered by G. Bennett [1] and W. Stadler [13], and later on was intensively studied in [6,7,8,9,10]. See also [12] for a related reference. Since we are dealing with Toeplitz duality here, let us consider the following extended version of the Wilansky property. An *FK*-space E containing Φ is said to have the Wilansky property $(\beta_T\text{-}W)$, if every dense *FK*-subspace F of E satisfying $F^{\beta_T} = E^{\beta_T}$ must coincide with E. So according to this terminology, *(W)* is now $(\beta\text{-}W)$.

The intention of this definition is well-understood in the light of our basic problem. Indeed, a space E having the Wilansky property $(\beta_T\text{-}W)$ is minimal with respect to its T-summability factor sequences $(E \to c_T)$, hence in particular is a candidate for being uniquely determined by these in an appropriate sense.

We are going to exhibit a nice class of spaces having the Wilansky property $(\beta_T\text{-}W)$, later on giving rise to the applications we have in mind. First, however, we need a preparation.

In [1, §4], Bennett also considered the γ-dual version of the Wilansky property *(W)*, profferring the name $(\gamma\text{-}W)$. When we transfer this concept to the Toeplitz sectional context, we obtain the properties $(\gamma_T\text{-}W)$, defined in the obvious way. But Bennett left open the question whether $(\gamma\text{-}W)$ is actually stronger than *(W)*. It turns out that this is *not* the case, i.e. $(\gamma\text{-}W)$ is only a reformulation of *(W)*, and we shall make use of this fact later on. We have the following

Proposition 2. *The Wilansky properties* $(\beta_T\text{-}W)$ *and* $(\gamma_T\text{-}W)$ *are equivalent for any FK-space E. In particular this is the case for the Wilansky properties* $(\beta\text{-}W)$ *and* $(\gamma\text{-}W)$.

Proof. Let E be an *FK*-space having property $(\gamma_T\text{-}W)$. Let F be a dense linear subspace of E satisfying $F^{\beta_T} = E^{\beta_T}$. As we already mentioned in the proof of Theorem 1, this equality implies the equaliy $F^{\gamma_T} = E^{\gamma_T}$, so $(\gamma_T\text{-}W)$ yields $F = E$, proving that E has property $(\beta_T\text{-}W)$. Conversely, assume that E has the Wilansky property $(\beta_T\text{-}W)$. Let F be a dense linear subspace of E satisfying $F^{\gamma_T} = E^{\gamma_T}$. We wish to prove $F^{\beta_T} = E^{\beta_T}$. The inclusion $E^{\beta_T} \subset F^{\beta_T}$ being clear, we fix $a \in F^{\beta_T}$. Let the $f_n \in E'$ be defined by

$$f_n(x) = \sum_{i=1}^{n} t_{ni} a_i x_i ,$$

then the sequence (f_n) is pointwise bounded in view of $a \in F^{\beta_T} \subset F^{\gamma_T} = E^{\gamma_T}$. On the other hand, $a \in F^{\beta_T}$ means that (f_n) pointwise converges on the dense linear subspace F of E. The Banach-Steinhaus Theorem therefore asserts that (f_n) converges on the whole space E, which just means $a \in E^{\beta_T}$. Now property $(\beta_T\text{-}W)$ applies and gives the equality $F = E$. This ends the proof. □

Before stating our next central result, we need another definition. Let E be a BK-space. We denote by R_E , or just by R if no confusion may occur, the norm closed linear hull of the projection functionals $x \to x_n$ in E'. So if Φ is considered part of E' via the natural identification, we just have $R_E = \overline{\Phi}$.

Theorem 3. *Let E be a BK-space containing Φ. Suppose R has a topological complement in E' and E^{γ_T} is separable. Then E has the Wilansky property $(\beta_T\text{-}W)$.*

Proof. As a consequence of Proposition 2 we have to show that every dense FK-subspace F of E satisfying $F^{\gamma_T} = E^{\gamma_T}$ must coincide with E. Following the argument in [1, Theorem 2], in order to do so, it suffices to show that any F of this kind is barrelled as a subspace of E. Moreover, when checking this, we may assume, as a consequence of the reasoning presented in the proof of Theorem 1 in [1], that F contains Φ. So let F be of this type. Let U be a barrel in F. We have to show that U is a neighbourhood of o with respect to the topology induced by E.

By assumption there exists a norm closed linear subspace Q of E' satisfying $E' = R \oplus Q$. Hence $E'' = R^{\perp} \oplus Q^{\perp} = Q' \oplus R'$. Let B be the polar of the unit ball of Q, calculated in the dual pairing $\langle R^{\perp},Q\rangle$. Then B is compact with respect to $\sigma(R^{\perp},Q) = \sigma(Q',Q)$, hence is as well compact with respect to $\sigma(E'',E')$ in view of the fact that $\sigma(E'',E')|R^{\perp} = \sigma(R^{\perp},Q)$.

Let $V = U + B$. Since the linear hull of V is $F + R^{\perp}$, the polar V° of V calculated with respect to the dual pairing $\langle E'',E'\rangle$, is bounded in $\sigma(E',F+R^{\perp})$. We prove that V° is as well norm bounded in E'.

Let (y^n) be a sequence in V°. Find sequences (r^n) in R , (q^n) in Q having $y^n = r^n + q^n$. Observe that (q^n) is bounded for $\sigma(Q,R^{\perp}) = \sigma(Q,Q')$. Indeed, for fixed $\psi \in R^{\perp}$ we have

$$\langle \psi, q^n \rangle \;=\; \langle \psi, r^n \rangle \;+\; \langle \psi, q^n \rangle \;=\; \langle \psi, y^n \rangle \;=\; O(1),$$

$n \to \infty$. As $\sigma(Q,R^{\perp}) = \sigma(Q,Q')$ is the weak topology corresponding with the dual norm, we deduce that (q^n) is also bounded with respect to the dual norm on Q. It therefore remains to prove that (r^n) is bounded in norm.

Recall that we have $R = \overline{\Phi}$ via identification. We therefore find a sequence (p^n) in Φ having $\|r^n - p^n\| \le 1$, $n \in \mathbb{N}$. As (r^n) is bounded for $\sigma(E',F)$, we deduce that (p^n) is bounded for $\sigma(\Phi,F)$. Applying Theorem 1, we find that $\sigma(\Phi,F)$ and $\sigma(\Phi,E)$ have the same null sequences, so they also have the same bounded sequences. Hence (p^n) is $\sigma(\Phi,E)$-bounded, proving that (r^n) is bounded in $\sigma(E',E)$. Using the Banach-Steinhaus Theorem, we finally obtain the norm boundedness of (r^n).

V° being bounded in norm, we deduce that $V^{\circ\circ}$ is a neighbourhood of o in E'', so $V^{\circ\circ} \cap F$ is a neighbourhood of o in F. But $V^{\circ\circ} = \overline{V} = \overline{U + B}$, where the closure refers to the topology $\sigma(E'',E')$. As B is $\sigma(E'',E')$-compact, $\overline{V} = \overline{U} + B$. Furthermore, $V^{\circ\circ} \cap F = \overline{V} \cap F = \overline{U} \cap F$ in view of $B \cap F = \{o\}$. Now observe that $\sigma(E'',E')|F = \sigma(F,E') = \sigma(F,F')$ and $U \subset F$. So $\overline{U} \cap F$ is the weak closure of U in F. But U was chosen weakly closed in F, so we have finally established the equality

$$V^{\circ\circ} \cap F \;=\; U,$$

which proves that U is a neighbourhood of o in F. □

Remark. The statement of Theorem 3 does no longer hold true if R is only assumed to have a quasi-complement in E', i.e. if there exists a closed linear subspace Q of E' such that $R \cap Q = \{o\}$ and $R + Q$ is dense in E'. For consider the example $E = \ell + q$, $E' = \ell_\infty \oplus q'$ with $T = S$, where q is the space of quasi-periodic sequences, i.e. the closure of the space of periodic sequences in ℓ_∞. E does not have the Wilansky property $(\beta\text{-}W)$: Take any dense proper FK-subspace D of ℓ, and let $F = D + q$. Then we have $F^\beta = E^\beta = \ell$. Nevertheless, R is separable (it always is), hence has a quasi-complement in $\ell_\infty \oplus q'$ (cf. [11]). Moreover $E^\gamma = \ell$ is separable here, so all other assumptions from Theorem 3 are met. For details concerning the space q we refer to reference [14A] of [14].

Applications. Let T be a lower triangular matrix with column limits 1. An FK-space E containing Φ is said to have T-sectional boundedness, TAB for short, if for fixed $x \in E$ the set $\{t^n x\colon n \in \mathbb{N}\}$ of T-sections of x is bounded in E.

Proposition 4. *Let E be a BK-space with TAB. Suppose E has separabel dual E' and R is complemented in E'. Then E has the Wilansky property $(\beta_T\text{-}W)$.*

Proof. Property *TAB* implies $E^{\gamma_T} = E^f$ (see [2]). But E^f is a quotient of E' under the natural mapping $f \to (f(e^n))$. As E' is separable, this proves that E has separable γ_T-dual. Hence Theorem 3 applies and gives the result. □

Remark. Proposition 4 implies as a special case the following result: Every *BK-TAK*-space E whose dual $E' = E^f = E^{\beta_T}$ is as well a *BK-TAK*-space, has the Wilansky property $(\beta_T\text{-}W)$. This is clear when we observe that *TAK* implies *TAB* (cf. [2]), and that the *TAK* assumption on E' means $R_E = E'$. So Proposition 4 in particular answers a question of Prof. G. Goes, who suggested that the original Bennett/Stadler result should carry over to the context of Toeplitz duality. It was pointed out to us by Prof. Goes that a student of his, U. Böttcher, has also obtained a Toeplitz version of the Bennett/Stadler result, using a different technique.

Proposition 5. *Let E be a BK-AD-space. Suppose R is complemented in E' and* E^{β_T} *is separable. Then E has the Wilansky property* $(\beta_T\text{-}W)$.

Proof. Property *AD* implies $E^{\beta_T} = E^{\gamma_T}$ (cf. [2, p.455]), hence E has separable γ_T-dual. Thus the prerequisites for applying Theorem 3 are met. □

Remark. Also Proposition 5 generalizes the original Bennett/Stadler result. For *TAK* implies *AD*, and *TAK* for $E' = E^{\beta_T}$ implies that the β_T-dual is separable.

We shall now present our first result concerning the reconstructability of a space from a corresponding space of factor sequences. In the case of β-duality, this was already obtained in [1,13].

Theorem 6. *Let E be a BK-TAK-space with separable* β_T*-dual. Suppose R is complemented in E'. Then E is uniquely determined by its space of T-summability factor sequences* $(E \to c_T)$ *among all FK-AD-spaces. In other terms, every FK-AD-space F having the same T-summability factors must coincide with E.*

Proof. Let F be an *FK-AD*-space satisfying $F^{\beta_T} = E^{\beta_T}$. Then we have $F^f \supset F^{\beta_T} = E^{\beta_T} = E^f$ (cf. [2]). Now we may apply the converse theorem for *f*-duals by Snyder/Wilansky ([14, 7.2.7]), which yields $F \subset E$. Then we apply Proposition 5 above, and this proves $F = E$. □

Our unicity theorem is optimal in a certain sense. Namely we have the following

Proposition 7. *Let E be a BK-AD-space which is uniquely determined by its T-summability factor sequences among all FK-AD-spaces. Then E must have TAK.*

Proof. Following [2, p.455], we have $E^{\beta_T} = E^{\gamma_T}$, and this space is *BK-TAB*. Also, Theorem 4 of [2] implies that $E^{\gamma_T f} = E^{\gamma_T \gamma_T}$. Setting $F = \left(E^{\gamma_T}\right)_{AD}$ in the notation used in [2], i.e. F is the closure of Φ in E^{γ_T}, it follows that F has *TAK* since *TAB* and *AD* together imply *TAK*. Using Proposition 1 of [2], it follows that $F' = F^f = E^{\gamma_T f} = E^{\gamma_T \gamma_T}$. This implies that $E^{\gamma_T \gamma_T}$ is *BK-TAB* by [2, Proposition 2]. Now let $G = \overline{\Phi}$ in $E^{\gamma_T \gamma_T}$. Then G is a *BK-TAK*-space containing E and satisfying $G^{\gamma_T} = E^{\gamma_T}$. For the latter statement compare [14, 10.3.23]. But now we may apply our assumption on E, which yields $G = E$. In particular this implies that E has *TAK*. □

Convergence domains. In this section we obtain applications of our abstract results in the context of convergence and summability domains. In particular, we derive results concerning the reconstructability of the Cesàro convergence domains and the Cesàro summability domains from appropriate spaces of factor sequences.

Theorem 8. *Let A be a perfect permanent lower triangular matrix with diagonal entries $\neq o$. Suppose $(c_A \to c_T)$ is separable, and A has T-summable sections (i.e. $(c_o)_A$ has TAK). Then c_A is uniquely determined by its T-summability factor space among the convergence domains of all perfect permanent methods.*

Proof. It suffices to observe that $E = (c_o)_A$ fulfills the requirements of Theorem 6. For if B is any perfect permanent method having the same set of summability factors $(c_B \to c_T)$, we set $F = (c_o)_B$, thus obtaining an *FK-AD*-space F satisfying $F^{\beta_T} = E^{\beta_T}$. So applying Theorem 6 gives $F = E$, and hence $c_B = c_A$.

Checking the conditions from Theorem 6, we first notice that the β_T-dual space $(c_o)_A^{\beta_T} = ((c_o)_A \to c_T) = (c_A \to c_T)$ is separable by assumption. Hence it remains to show that R_E coincides with E' here. But recall that E' may be identified with ℓ in the usual way. In view of perfectness, this identification naturally maps the linear hull L of the projection functionals $x \to x_n$ in E' onto the dense subspace Φ of ℓ. Hence we obtain the desired relation $R_E = E'$. □

In particular, Theorem 8 tells that a convergence domain c_A having sectional convergence is uniquely determined by its summability factors among all perfect methods.

Corollary 9. *For $0 \le \alpha \le 1$, the convergence domain of the Cesaro method C_α is uniquely determined by its set of summability factors $(c_{C_\alpha} \to c_S)$ among the convergence domains of all perfect permanent methods. For $\alpha > 1$, the convergence domain of the Cesaro method C_α is uniquely determined by its set of $C_{\alpha-1}$-summability factor sequences $(c_{C_\alpha} \to c_{C_{\alpha-1}S})$ among the convergence domains of all perfect permanent methods.*

Proof. Following [15, S. 104], the methods C_α, $0 \le \alpha \le 1$, have sectional convergence. So in this case Theorem 8 above gives the result when we take $T = S$.

For $\alpha > 1$, the methods C_α are known to have $C_{\alpha-1}$-summable sections (cf. [5]). Hence Theorem 8 applies again with the choice $T = C_{\alpha-1}S$. □

Corollary 10. *For $\alpha > 0$, the summability domain of the Cesaro method C_α is uniquely determined by its C_α-summability factor sequences among the summability domains of all perfect permanent methods.*

Proof. Following [5], the summability domains $E = c_{C_\alpha S}$ have C_α-summable sections, i.e. they are *TAK*-spaces in our present terminology, where $T = C_\alpha S$. Using the argument from the proof of Theorem 8, we see that $R_E = E'$ holds as well, so Proposition 6 applies and gives the result. □

We end this section with the following result concerning the minimality of a perfect convergence domain c_A with respect to a corresponding space of T-summability factor sequences $(c_A \to c_T)$.

Theorem 11. *Let A be a perfect permanent lower triangular matrix with diagonal entries $\neq 0$. Suppose $(c_A \to c_T)$ is separable. Then c_A is minimal with respect to its T-summability factor sequences among the convergence domains of all permanent methods.*

Proof. We have to show that $c_A^{\beta_T} = c_B^{\beta_T}$ implies $c_A = c_B$ for every permanent method B having $c_B \subset c_A$. This is a consequence of the fact that, by Proposition 5, $E = (c_o)_A$ has the Wilansky property $(\beta_T\text{-}W)$. Indeed, setting $F = (c_o)_B$ provides a dense FK-subspace of E satisfying $F^{\beta_T} = ((c_o)_B \to c_T) = (c_B \to c_T) = (c_A \to c_T) = ((c_o)_A \to c_T) = E^{\beta_T}$. This implies $F = E$, hence $c_B = c_A$.□

Concluding remarks. Let T,R be lower triangular matrices with column limits 1. One may ask under what conditions the Wilansky property $(\beta_T\text{-}W)$ implies the Wilansky property $(\beta_R\text{-}W)$ for a space E. This seems to be a difficult point. We have the following partial answer.

Proposition 12. *Let E be a BK-space containing Φ. Suppose E has the Wilansky property $(\beta_R\text{-}W)$, and E^{γ_T} is separable. Then E has the Wilansky property $(\beta_T\text{-}W)$.*

Proof. Let F be a dense FK-subspace of E satisfying $F^{\beta_T} = E^{\beta_T}$. Then Theorem 1 applies in view of the separability of E^{γ_T}. Hence condition (5) from Theorem 1 is valid. Using the argument from the proof of Theorem 1, this implies $F^{\beta_R} = E^{\beta_R}$. So property $(\beta_R\text{-}W)$ applies and gives $F = E$. □

This result naturally raises the following question. Given a BK-space E, under what conditions does there exist a matrix T such that E^{γ_T} is separable. In particular, is it possible to exhibit such T for every BK-AD-space? Or even more specially, can such T always be provided in the case of a perfect convergence domain c_A?

Our present results do not tell us whether the Cesàro convergence domains c_{C_α}, $\alpha > 1$, or the Cesàro summability domains $c_{C_\alpha S}$, $\alpha > 0$, have the Wilansky property $(\beta\text{-}W)$, since the corresponding β-dual spaces are not separable here. Actually, we do not know even whether the Cesàro convergence domain c_{C_2} and the Cesàro summability domain $c_{C_1 S}$ are spaces having the Wilansky property (W). It should be expected, however, that they are, in particular, that the domains c_{C_α}, (and similarly the $c_{C_\alpha S}$) are uniquely determined by their summability factor sequences among all perfect permanent convergence resp. summability domains. But certainly they are not determined by their summability factor spaces among *all* spaces of the form $E + lin\{e\}$, where E is any BK-AD-space. Indeed, by Proposition 7, the latter would imply that the C_α had sectional convergence, which is not the case for $\alpha > 1$. Generalizing this situation, we may state the following problem:

Let A be a perfect and permanent summability method such that c_A is uniquely determined by $(c_A \to c_T)$ among the convergence domains of all perfect permanent methods. Must A have T-summable sections? In particular, if c_A is uniquely determined by $(c_A \to c_S)$, must A have sectional convergence?

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