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WEAKENING TAUBERIAN CONDITIONS FOR SUMMABILITY METHODS

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ABSTRACT. We discuss Tauberian conditions for summability methods of the type (i) "$\Sigma(-1)^n t_n x_n$ converges", (ii) "$t_n x_n = o(1)$" and (iii) "$\frac{1}{n}\sum_{i=1}^{n} t_i x_i = o(1)$". Using functional analytic methods, we obtain conditions on the sequence (t_n) under which (i) and (ii), (ii) and (iii) are equivalent Tauberian conditions for every linear and permanent summability method. These results depend on the fact that c_o and $(c_o)_{C_1}$ are spaces with the Wilansky property.

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Introduction

In this paper we deal with the problem of weakening Tauberian conditions for summability methods. Apparently, the first result of this kind was obtained by Meyer-König and Tietz [8], who proved that whenever "$nx_n = o(1)$" is a Tauberian condition for some linear and permanent summability method V, then the weaker condition "$\frac{1}{n}\sum_{i=1}^{n} ix_i = o(1)$" is as well Tauberian for V. Various other results of this type, dealing with related problems, are known. We just mention [2], [3], [4], [5], [6], [7], [8], [10], [12], [13].

For the first time, Goes ([3], [4]) brought up functional analytic methods in the treatment of such kind of problems. Our present attempt as well is a functional analytic one. We prove that a Tauberian condition "$x \in E$" may always be weakened to "$x \in F$" if and only if $F \subset E + cs$. In many cases, however, the latter inclusion may be replaced by the condition $E^\beta \cap bv \subset F^\beta$. When this replacement is possible, is decided by functional analytic methods. The gain of this reduction is that $E^\beta \cap bv \subset F^\beta$ seems to be better adapted to analytic treatment.

We try our method on the following Tauberian conditions

(i) $\sum_{n=1}^{\infty} (-1)^n t_n x_n$ converges,

(ii) $t_n x_n = o(1)$, $(n \to \infty)$,

(iii) $\frac{1}{n} \sum_{i=1}^{n} t_i x_i = o(1)$, $(n \to \infty)$,

asking for which sequences (t_n) conditions (i) and (ii) resp. (ii) and (iii) are equivalent for every linear, permanent summability method V.

Notation and preliminary results

Concerning notations from FK - space theory, we refer to [14]. Summability is covered by [15].

An FK - space E is said to have the Wilansky property (see [1]) if every subspace F of E with $F^\beta = E^\beta$ is barrelled in E, or equivalently, if every FK - subspace F of E satisfying $F^\beta = E^\beta$ is closed in E. It was proved in [1] and [11] that c_0 has the Wilansky property. Here we shall need the following more general

THEOREM 0. ([9, theorem 2]).
Let E be a BK - AK - space such that S(E'), the space of all sequences $y \in E'$ having sectional convergence with respect to the norm, is separably complemented in E'. Then E has the Wilansky property. □

General results

Let D be any linear subspace of ω containing cs and let $V: D \to \mathbb{C}$ be a linear mapping satisfying $V(x) = \sum x_i$ for $x \in cs$. Then V is called a linear, permanent summability method (or just a summability method). Examples of particular interest are matrix summability methods. Here we have $V(x) = A\text{-}\Sigma x_i$ for some permanent infinite matrix A.

Let E be a sequence space. Then "$x \in E$" is called a Tauberian condition for the summability method V with domain D if $D \cap E \subset cs$. The following easy result, our starting point, gives a description of when a certain Tauberian condition "$x \in E$" may be replaced by a (usually formally weaker) condition "$x \in F$".

<u>PROPOSITION 1</u>. Let E,F be sequence spaces. Then the following statements are equivalent:

(1) Whenever "$x \in E$" is a Tauberian condition for some linear, permanent summability method V, then also "$x \in F$" is a Tauberian condition for V;

(2) $F \subset E + cs$.

<u>Proof</u>. Assume (1). Suppose $F \not\subset E + cs$, let $y \in F$, $y \notin E + cs$. Let $D = cs + \text{lin}\{y\}$ and define $V:D \to \mathbb{C}$ by $V(y) = o$, $V(x) = \Sigma x_i$ for $x \in cs$. Clearly "$x \in E$" is Tauberian for V but "$x \in F$" is not, a contradiction.

Assume (2). Let V be a summability method such that "$x \in E$" is Tauberian for V. Let $x \in D \cap F$. Then $x = y + z$, $y \in E$, $z \in cs$. Since $D \supset cs$, $y \in D$, hence $y \in cs$ since "$x \in E$" is Tauberian for V. This gives $x \in cs$. □

It seems to be a considerably more difficult task to replace statement (1) above by the corresponding phrase referring to matrix summability methods only. No general result of the above type seems to be valid, but we have a partial answer.

<u>PROPOSITION 2</u>. Let E,F be sequence spaces. Suppose F admits a decomposition $F = F_+ - F_+$, where F_+ denotes the cone of all $y \in F$ with $y_n \geqslant o$ for all n. Then the following are equivalent:

(1) Whenever "$x \in E$" is a Tauberian condition for some permanent matrix summability method $V = A\text{-}\Sigma$, the same is true for the condition "$x \in F$";

(2) $F \subset E + cs$.

<u>Proof</u>. (2) implies (1) by proposition 1. Assume $F \not\subset E + cs$. Then $F_+ \not\subset E + cs$. Choose $y \in F_+$, $y \notin E + cs$. Notice that Sy =

$(y_1, y_1+y_2, y_1+y_2+y_3, \ldots)$ must be unbounded. Now a classical result of Mazur (see [14, p.48]) asserts the existence of a permanent matrix A with domain $c_A = c + \mathrm{lin}\{Sy\}$. Consequently, $V = A-\Sigma$ has domain $D = cs + \mathrm{lin}\{y\}$. Again, "$x \in E$" is Tauberian for V, but "$x \in F$" is not. □

Wilansky property enters

It is clear that a necessary condition for the validity of the inclusion $F \subset E + cs$ is $E^\beta \cap bv \subset F^\beta$. It turns out that, under certain circumstances, the latter condition is also sufficient to imply $F \subset E + cs$.

THEOREM 1. Let E be an FK - space with FAK and let F be an FK-AK - space with the Wilansky property. Then $F \subset E + cs$ if and only if $E^\beta \cap bv \subset F^\beta$.

Proof. Clearly $F \subset E + cs$ implies $F^\beta \supset (E + cs)^\beta = E^\beta \cap cs^\beta = E^\beta \cap bv$. Conversely, assume $E^\beta \cap bv \subset F^\beta$. We have to prove that $(E + cs) \cap F = F$. Let $G = (E + cs) \cap F$. Then G is an FK - space with its 'natural' topology (see [15, pp.59, 221]) and is dense in F in view of the fact that it contains Φ. Since F has the Wilansky property, it suffices to prove $G^\beta = F^\beta$, for then G is closed in F, and this implies $G = F$.

We have $G^\beta = ((F \cap E) + (F \cap cs))^\beta = (F \cap E)^\beta \cap (F \cap cs)^\beta$. Since E has FAK and F, cs have AK, [4, Satz 2.3] implies that $(F \cap E)^\beta = F^\beta + E^\beta$, $(F \cap cs)^\beta = F^\beta + cs^\beta = F^\beta + bv$. This yields $G^\beta = (F^\beta + E^\beta) \cap (F^\beta + bv) = F^\beta + (E^\beta \cap bv) = F^\beta$. This ends the proof. □

When it is known that $E \subset F$, a similar result may be obtained without the FAK - condition on the space E.

THEOREM 2. Let E be an FK - space and let F be a BK - AK - space such that $S(F')$ has a separable complement in F'. Suppose $E \subset F$. Then $E^\beta \cap bv \subset F^\beta$ implies $F \subset E + cs$.

Proof. We have to prove $F + cs = E + cs$. First observe that the space $F + cs$ has the Wilansky property. Indeed, $F + cs$ is an AK - space, hence $(F + cs)' = (F + cs)^\beta = F^\beta \cap bv = F' \cap bv$. But $S(F' \cap bv) = S(F') \cap bv_o$, where $bv_o = bv \cap c_o$. Suppose $F' = S(F')$

$\oplus$ L, then $F' \cap bv = (S(F') \cap bv) \oplus (L \cap bv) = (S(F') \cap bv_o) \oplus (S(F') \cap \mathrm{lin}\{e\}) \oplus (L \cap bv)$, hence $S(F' \cap bv)$ is complemented in $F' \cap bv$ with separable complement $(S(F') \cap \mathrm{lin}\{e\}) \oplus (L \cap bv)$. By theorem o, F + cs has the Wilansky property.

Observe that E + cs is a dense subspace of F + cs. Since the latter space has the Wilansky property, it suffices to prove that $(E + cs)^\beta = (F + cs)^\beta$. But this is clear in view of our assumption: $(E + cs)^\beta = E^\beta \cap bv \subset F^\beta$, hence $(E + cs)^\beta \subset (F + cs)^\beta$. The reverse inclusion is obvious. □

Weakening (i) to (ii)

We shall now apply our results from the previous section to the problem of weakening condition (i) to (ii). In the case $t_n = n$, this problem has been treated by Goes in [3]. Here we assume that (t_n) is any sequence having $t_n \neq o$ for all n.

Let $E = \{x \in \omega: ((-1)^n t_n x_n) \in cs\}$, $F = \{x \in \omega: (t_n x_n) \in c_o\}$. Then condition (i) is "$x \in E$", condition (ii) is "$x \in F$". Note that E,F are both BK - spaces with their natural topologies. For E take the topology inhereted from cs, for F the topology coming from c_o. Since F is a domain of c_o with respect to a diagonal matrix, the Bennett/Stadler result ([1], [11]) shows that the space F has the Wilansky property. We may therefore apply theorem 1 or 2. This gives the following

PROPOSITION 3. Let (t_n) with $t_n \neq o$ be fixed. Then the following statements are equivalent:

(1) Whenever (i) is a Tauberian condition for some permanent matrix summability method $V = A\text{-}\Sigma$, then (ii) as well is a Tauberian condition for V;

(2) Whenever $z \in bv$ satisfies $((-1)^n z_n/t_n) \in bv$, then it must in fact satisfy $(z_n/t_n) \in l^1$.

Proof. By proposition 2, statement (1) above is equivalent to $F \subset E + cs$, where E,F are defined as above. By theorem 1 or 2, this is equivalent to $E^\beta \cap bv \subset F^\beta$. By the definition of E, F,

$$E^\beta = \{z \in \omega: ((-1)^n z_n/t_n) \in bv\},$$
$$F^\beta = \{z \in \omega: (z_n/t_n) \in l^1\},$$

and so the inclusion readily implies statement (2). □

<u>THEOREM 3</u>. Let (t_n) be any sequence of reals having $t_n \neq o$. Then the following conditions (1) or (2) are sufficient for the equivalency of (i) and (ii) as Tauberian conditions:
(1) $(1/t_n) \in bv$, (2) $t_n \geqslant \delta > o$ for all n.

<u>Proof</u>. In both cases we check statement (2) of proposition 3. First consider case (1). Let $z \in bv$ satisfying $((-1)^n z_n/t_n) \in bv$ be fixed. Since $bv \cdot bv \subset bv$, condition (1) implies $(z_n/t_n) \in bv$. Combining this with the above condition implies

$$(o, z_2/t_2, o, z_4/t_4, o, \ldots) \in bv,$$

hence $(z_{2n}/t_{2n}) \in l^1$. Consequently, we also obtain

$$(z_1/t_1, o, z_3/t_3, o, \ldots) \in bv,$$

giving $(z_{2n+1}/t_{2n+1}) \in l^1$, hence $(z_n/t_n) \in l^1$.

Let us now consider case (2). Let $z \in bv$ with $((-1)^n z_n/t_n) \in bv$ be fixed. The latter may be expressed by

$$(z_n/t_n + z_{n+1}/t_{n+1}) \in l^1.$$

But note that

$$z_n/t_n + z_{n+1}/t_{n+1} = (z_n - z_{n+1})/t_n + z_{n+1}(1/t_{n+1} + 1/t_n).$$

Since $z \in bv$ implies $(z_n - z_{n+1}) \in l^1$, (2) gives

$$((z_n - z_{n+1})/t_n) \in l^1.$$

Therefore, $(z_{n+1}(1/t_{n+1} + 1/t_n)) \in l^1$, hence $(|z_{n+1}|(1/t_{n+1} + 1/t_n)) \in l^1$. Since

$$o \leqslant |z_{n+1}|(1/t_{n+1}) \leqslant |z_{n+1}|(1/t_{n+1} + 1/t_n),$$

we obtain the desired $(z_n/t_n) \in l^1$. □

From a practical point of view, theorem 3 covers all interesting cases of sequences (t_n) giving rise to Tauberian conditions (i) or (ii). Nevertheless, one may ask whether (i) and (ii) are still equivalent Tauberian conditions in the case where the sequence (t_n) is not bounded away from o. Here we have the following partial answer.

<u>PROPOSITION 4</u>. Let (t_n) be a positive null sequence. Then

there exists a permanent matrix summability method $V = A\text{-}\Sigma$ such that (i) is a Tauberian condition for V, but (ii) is not.

Proof. We prove that statement (2) of proposition 3 is not valid here. Let T denote the diagonal matrix with entries $t_1, t_2, \ldots$ and let R denote the diagonal matrix with entries $1,-1,1,-1,\ldots$. Then statement (2) of proposition 3 may be translated into

$$bv \cap bv_{T^{-1}R} \subset l^1_{T^{-1}}.$$

Clearly, $bv \cap bv_{T^{-1}R} = (c_S + c_{STR})^\beta$, where $c_S = cs$ and where S is the summation matrix. But

$$(c_S + c_{STR})^{\beta\beta} \supset (l^1_{T^{-1}})^\beta = m_T,$$

giving

$$((c_{ST^{-1}} + c_{SR})_T)^{\beta\beta} \supset m_T$$

hence

$$(c_{ST^{-1}} + c_{SR})^{\beta\beta} \supset m.$$

But this is possible only in the case where $H = c_{ST^{-1}} + c_{SR}$ contains c_o. Indeed, H is an FK - space densely contained in c_o: We have $\Phi \subset c_{SR} \subset c_o$ and $c_{ST^{-1}} \subset c_o$, the latter since $t_n \to o$. But the above calculation shows $H^\beta = c_o{}^\beta = l^1$, hence $H = c_o$ as a consequence of the fact that c_o has the Wilansky property. Thus $c_o = c_{ST^{-1}} + c_{SR}$. Since $c_{ST^{-1}} \subset (c_o)_{T^{-1}} \subset c_o$, we obtain

$$c_o = (c_o)_{T^{-1}} + c_{SR},$$

which in view of $(c_o)_R = c_o$ and $((c_o)_{T^{-1}})_R = (c_o)_{T^{-1}}$ gives us

$$c_o = (c_o)_{T^{-1}} + c_S.$$

It is easy to see, however, that the latter equality is not true in the case $t_n \to o$, so statement (2) cannot be true either. □

Remarks. 1) Theorem 3 (1) has been proved by the second author in [12] by a different method. The case $t_n = n$ has been treated by Goes [3], Buntinas [2] and Kuttner/Parameswaran [6]. In [2], similar problems are considered where the sequence $(-1)^n$ is replaced by other sequences of ±1 entries.

2) Theorem 3 (2) may still be generalized to some extent. We may

assume that the sequence (t_n) is divided into blocks I_1, J_1, I_2, $J_2, \ldots$, where $t_n \geqslant \delta > 0$ on $I_1 \cup I_2 \cup \ldots$, $t_n \to 0$ on $J_1 \cup J_2 \cup \ldots$. Suppose the blocks J_i are of bounded length $|J_i| \leqslant r$. Then (i) and (ii) are still equivalent Tauberian conditions. This may be shown using a similar reasoning as in the proof of theorem 3(2).

Weakening (ii) to (iii)

We shall now examine the problem of replacing the Tauberian condition (ii) by the weaker condition (iii). We shall use the same method.

Let F be defined as in the previous section and let $G = \{x \in \omega: (t_n x_n) \in (c_o)_{C_1}\}$, where $(c_o)_{C_1}$ denotes the zero domain of the Cesàro method C_1. Then condition (iii) is "$x \in G$". Asking for conditions on the sequence (t_n) under which (ii) may be weakened to (iii) therefore leads to the question when $G \subset F + cs$. Again we wish to express this by $F^\beta \cap bv \subset G^\beta$. This requires checking the assumptions of theorem 1. Recall that F is a BK - AK - space. On the other hand, G is a BK - space since $(c_o)_{C_1}$ is.

LEMMA. G is a BK - AK - space with the Wilansky property.

Proof. First observe that $(c_o)_{C_1}$ is a BK - AK - space (see [15, p.42 or p.104]). Its dual and its[1] β-dual therefore coincide. But

$$(c_o)_{C_1}^\beta = \{z \in \omega: \sum_{n=1}^{\infty} n|z_n - z_{n+1}| < \infty,\ nz_n = O(1)\}.$$

This may in fact be derived from [15, p.105]. We wish to prove that this β-dual has sectional convergence, which is not clear from the above form. But note that we may add on the right hand side the redundant condition $z \in l^1$. Using the identity

$$n(z_n - z_{n+1}) = (nz_n - (n+1)z_{n+1}) + z_{n+1},$$

this shows that

$$(c_o)_{C_1}^\beta = \{z \in \omega: (nz_n) \in bv,\ z \in l^1\}.$$

Let us prove that we may actually write $(nz_n) \in bv_o$ on the right hand side here. Indeed, suppose we had $nz_n \to 1$ for some z in the β-dual. This means $|z_n| \geqslant 1/2n$ eventually, contradicting $z \in l^1$.

This proves that $(c_o)_{C_1}^{\beta} = (c_o)_{C_1}'$ is a BK - AK - space, hence the result of Bennett [1] and Stadler [11] implies that $(c_o)_{C_1}$ has the Wilansky property. Since G is the domain of $(c_o)_{C_1}$ with respect to a diagonal matrix, we deduce that G is as well a BK - AK - space whose dual G' is BK - AK, hence G has the Wilansky property. □

Notice that by the above calculation of $(c_o)_{C_1}^{\beta}$, the β-dual of G is

$$G^{\beta} = \{z \in \omega: (nz_n/t_n) \in bv_o, (z_n/t_n) \in l^1\}.$$

This permits us to state the following

<u>PROPOSITION 5</u>. Let (t_n) be a fixed sequence having $t_n \neq o$ for all n. The following statements are equivalent:

(1) Whenever (ii) is a Tauberian condition for some linear, permanent summability method V, then (iii) as well is Tauberian for V;

(2) Whenever $z \in bv$ satisfies $(z_n/t_n) \in l^1$, then it must in fact satisfy $(nz_n/t_n) \in bv$.

<u>Proof</u>. The proof is now clear in view of proposition 1, theorem 1, the lemma above, and the above calculation of F^{β} and G^{β}.□

<u>THEOREM 4</u>. Let (t_n) with $t_n \neq o$ be fixed. A sufficient condition for the equivalence of (ii) and (iii) as Tauberian conditions is $(n/t_n) \in bv$. In particular, this is the case when (n/t_n) is bounded and monotone.

<u>Proof</u>. We check condition (2) of proposition 5. But this is clear here, since $z \in bv$, $(n/t_n) \in bv$ implies $(nz_n/t_n) \in bv$ in view of the inclusion $bv \cdot bv \subset bv$. The second part of the result is clear since boundedness and monotonicy of (n/t_n) gives us $(n/t_n) \in bv$. □

The second part of theorem 4 is [8, Satz 2.1]. We obtain another result from [8] using proposition 5.

<u>THEOREM 5</u>. If (t_n) with $t_n \neq o$ satisfies $n/t_n = O(1)$ and $n(t_{n+1} - t_n) = O(t_{n+1})$, then (ii) may be weakened to (iii).

Proof. We establish (2) from proposition 5. Let $z \in bv$ having $(z_n/t_n) \in l^1$ be fixed. Since $(n(t_{n+1}-t_n)/t_{n+1})$ is bounded, we obtain

$$(nz_n/t_n - nz_n/t_{n+1}) \in l^1.$$

By assumption, $n/t_{n+1} = (n+1)/t_{n+1} \cdot n/(n+1) = O(1)$, so $z \in bv$ implies

$$(n/t_{n+1})(z_n - z_{n+1}) \in l^1.$$

In view of the identity

$$nz_n/t_n - (n+1)z_{n+1}/t_{n+1} = nz_n/t_n - nz_n/t_{n+1} + (n/t_{n+1})(z_n - z_{n+1}),$$

this implies $(nz_n/t_n) \in bv$. □

The authors of [8] prove that in the case where $n/t_n \to \infty$, (ii) and (iii) are not equivalent Tauberian conditions. In fact, they prove that in this case there exists a permanent matrix method $V = A \circ S$ such that (ii) is a Tauberian condition for V, but (iii) is not. Using our proposition 5, we obtain the following related

PROPOSITION 6. Let (t_n) with $t_n \neq o$ be fixed. Suppose that (iii) is a Tauberian condition for a summability method V whenever this is true for (ii). Then $n/t_n = O(1)$ and $(n^\alpha/t_n) \in bv$ for α having $o \leqslant \alpha < 1$.

Proof. We use condition (2) of proposition 5. First we prove that $|t_n| \geqslant \delta > o$. Assume the contrary and let $t_{n_i} \to o$ for some sequence (n_i) of indices satisfying $n_{i+1} - n_i \geqslant 2$. First assume that $\Sigma 1/n_i$ converges. Define $z \in l^1$ by setting $z_n = t_n/n$ in case $n = n_i$, $z_n = o$ otherwise. Since $(z_n/t_n) \in l^1$, we have $(nz_n/t_n) \in bv$. But this is absurd. Next assume that $\Sigma 1/n_i = \infty$. Here we set $z_n = t_n/n^2$ in case $n = n_i$, $z_n = o$ otherwise. Again this implies $(nz_n/t_n) \in bv$ in view of condition (2) of proposition 5, a contradiction. This proves $|t_n| \geqslant \delta > o$.

Since $(1/t_n)$ is bounded, every sequence $z \in l^1$ may now be taken as a test sequence in (2) of proposition 5. Hence $(nz_n/t_n) \in bv$ holds for all $z \in l^1$. This proves $n/t_n = O(1)$. Indeed, (n/t_n) is an element of $\{y \in \omega: yz \in bv \text{ for all } z \in l^1\}$, which is just m.

Finally, let $o \leqslant \alpha < 1$, $z_n = 1/n^{1-\alpha}$, then $z \in bv$, $(z_n/t_n) \in l^1$. Therefore, $(nz_n/t_n) = (n^\alpha/t_n) \in bv$, as desired. □

<u>Remark</u>. We do not know whether proposition 6 also holds in the case $\alpha = 1$, i.e. whether $(n/t_n) \in bv$ is both a necessary and sufficient condition for the replacability of (ii) by (iii).

<u>Concluding remarks</u>

Our present methods also apply in the case where the sequence (t_n) has entries o. Indeed, in this case let (s_n) denote the sequence arising from (t_n) by omitting all entries $t_n = o$. Now let $E_o = \{x \in \omega: (\sigma_n s_n x_n) \in cs\}$, $F_o = \{x \in \omega: (s_n x_n) \in c_o\}$, $G_o = \{x \in \omega: (s_n x_n) \in (c_o)_{C_1}\}$, where $s_n = t_m$ implies $\sigma_n = (-1)^m$. Then (i), (ii), (iii) are described by "$\bar{x} \in E_o$", "$\bar{x} \in F_o$", "$\bar{x} \in G_o$" respectively, when $\bar{x}$ denotes the sequence arising from x by discarding the entries x_m where $t_m = o$. Clearly, the equivalency of (i) and (ii) may now be expressed by $F_o \subset E_o + cs$, while the replacability of (ii) by (iii) turns into $G_o \subset F_o + cs$. This transcription permits us to apply once more our functional analytic approach.

One may consider Tauberian conditions of type (iii), where the Cesàro method is replaced by some permanent method of weighted means. This provides Tauberian conditions of type (iii) also in the case where the sequence (t_n) does not satisfy condition (2) of proposition 5. For instance, the Borel method B has Tauberian condition "$\sqrt{n}x_n = o(1)$". Since $(n^{3/4}/n^{1/2})$ is not in bv, we cannot weaken this o-condition to "$\frac{1}{n}\sum_{i=1}^{n}\sqrt{i}x_i = o(1)$". And, in fact, it is known that the latter condition is no longer Tauberian for the method B (see [8]). Nevertheless, it is possible to obtain (iii) type Tauberian conditions for the Borel method by replacing C_1 by an appropriate method of weighted means.

Absolute Tauberian conditions may as well be treated using our present methods. The result corresponding with proposition 1 in this case states that an absolute Tauberian condition "$x \in E$" for some linear, permanent absolute summability method V may be replaced by an absolute Tauberian condition "$x \in F$" if and only if $F \subset E + l^1$. Depending on the spaces E,F, the latter condition ought to be replaced by the dual condition $E^\beta \cap m \subset F^\beta$, which should be expected to be somewhat better adapted to further analytic treatment.

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